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COEFFICIENT BOUNDS FOR A NEW CLASS OF UNIVALENT FUNCTION BY CHEBYSHEV POLYNOMIAL

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***Abstract*—In this work, the class $\mathcal{M}_D(\tau, t)$ of a normalized analytic univalent function defined by the subordination of Chebyshev polynomial of the second kind was introduced. The initial coefficients for the class was obtained. Furthermore, the Fekete-Szego functional involving the class was examined.**

***Index Terms*—analytic functions, univalent function, subordination, Chebyshev polynomial, Fekete-Szegö functional.**

1. INTRODUCTION

Let A represent the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad (z \in \mathbb{U}) \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ normalised by $f(0) = 0$ and $f'(0) = 1$. Also S represent the class of analytic univalent and normalised function in $\mathbb{U}$.

The function l is said to be subordinate to L, written $l(z) \prec L(z)$, if there exists a function ω analytic in $\mathbb{U}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ and such that $l(z) = L(\omega(z))$.

The Chebyshev polynomials are of four kinds. Among these four kinds, the first and second kinds $T_n(t)$ and $U_n(t)$ are mostly used in geometric function theory. Najafzadeh and Salleh (2022) determined the coefficient bound and convolution property of a new class of univalent functions associated with Chebyshev polynomial.

The Chebyshev polymomial of the first and second kind are defined respectively as;

$$T_n(t) = \cos n\alpha$$

$$U_n(t) = \frac{\sin(n+1)\alpha}{\sin\alpha} \qquad t \in (-1, 1),$$

where $t = \cos\alpha$ and n denotes the degree of the polynomial.The generating functions of both the first and second kind of Chebyshev polynomials are defined respectively by:

$$\sum_{n=0}^{\infty} T_n(t) z^n = \frac{1 - tz}{1 - 2tz + z^2}$$

$$H(z, t) = \frac{1}{1 - 2tz + z^2} = 1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\alpha}{\sin\alpha} z^n$$

Then from Whittaker and Watson (1963);

$$H(z, t) = 1 + U_1(t) + U_2(t)z^2 + U_3(t)z^3 + \cdots \tag{1.2}$$

Let

$$U_{n-1} = \frac{\sin n \arccos t}{\sqrt{1 - t^2}} \qquad (n \in \mathbb{N},)$$

Then first and second kind are connected by the relation.

$$\begin{aligned} T_n(t) &= U_n(t) - tU_{n-1}(t) \\ U_n(t) &= 2tU_{n-1}(t) - U_{n-2}(t) \end{aligned} \tag{1.3}$$

then;

$$\begin{aligned} U_1(t) &= 2t, \\ U_2(t) &= 4t^2 - 1, \\ U_3(t) &= 8t^3 - 4t \end{aligned} \tag{1.4}$$

Various works have been done with Chebyshev polynomials in literature. [5] [11] [13].

In geometric function theory, one of the typical problem is the study of a functionals that consists of combinations of the coefficients of the initial function. Most often, there is a parameter for which the function extremal value is required. It is defined by the coefficient of the odd number functions.

$$l(z) = \sqrt{f(z)^2} = z + e_3 z^3 + e_5^5 + \cdots$$

In 1933, Fekete-Szego proved that;

$$|a_3 - \sigma a_2^2| \leq \begin{cases} 3 - 4\sigma & \textit{if } \sigma \leq 0 \\ 1 + e\left(\frac{-2\sigma}{1-\sigma}\right) & \textit{if } 0 \leq \sigma \leq 1 \\ 4\sigma - 3 & \textit{if } \sigma \geq 1 \end{cases} \tag{1.5}$$

The determination of the sharp bounds for the functional $|a_3 - \sigma a_2^2|$ is known as the Fekete-Szego functional.

Hari et.al (2020) studied a new subclass of normalized analytic

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functions with respect to symmetrical points in an open unit disk.The Fekete Szego functional has been investigated by several authors in literature. [8] [15].

2. PRELIMINARIES

Altinkaya and Sibel (2016) [13], Fadipe-Joseph et.al (2017) [4] and Yasamian et.al (2020) [18] established the coefficient bounds for different classes of analytic univalent functions. Yasamian et.al (2020) [18] introduced a superclass $\mathcal{M}_D$ which is defined as:

$$\mathcal{M}_D :\equiv \left\{ f \in \mathcal{H}(\mathbb{D}) : 2f\left(\frac{z}{2-z}\right) \in S \right\}.$$

From (2),

$$g(z) = 2f\left(\frac{z}{2-z}\right).$$

It follows that

$$g(z) = 2\left(\frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \frac{z^4}{16} + \frac{z^5}{32} + \cdots\right) + 2\sum_{k=2}^{\infty} a_k \left(\frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \frac{z^4}{16} + \frac{z^5}{32} + \cdots\right)^k.$$

Then by expanding and taking like terms we obtain;

$$g(z) = z + \left(\frac{1}{2} + \frac{a_2}{2}\right)z^2 + \left(\frac{1}{4} + \frac{a_2}{2} + \frac{a_3}{4}\right)z^3 + \left(\frac{1}{8} + \frac{3a_2}{8} + \frac{3a_3}{8} + \frac{a_4}{8}\right)z^4 + \left(\frac{1}{16} + \frac{a_2}{4} + \frac{3a_3}{8} + \frac{a_4}{4} + \frac{a_5}{16}\right)z^5 + \cdots \tag{2.1}$$

Lemma 2.1. *([3]) Let ω be an analytic function then from the principle of subordination, it shows that if $|\omega(z)| = |e_1 z + e_2 z^2 + e_3 z^3 + e_4 z^4 + \cdots| < 1, \quad z \in \mathcal{H}$, then*

$$|e_i| \leq 1, \qquad \textit{for all} \quad (i \in \mathbb{N}) \tag{2.2}$$

and

$$|e_2 - \psi e_1^2| \leq \max\{1, |\psi|\} \quad (\psi \in \mathbb{R}) \tag{2.3}$$

Definition 2.2. *The function $g \in \mathcal{M}_D$ is in the class $\mathcal{M}_D(\tau, t)$ where $0 \leq \tau \leq 1$ and $t \in \left(\frac{1}{2}, 1\right)$ if the following subordination holds*

$$(1-\tau)\frac{zg'(z)}{g(z)} + \tau\left(1 + \frac{zg''(z)}{g'(z)}\right) \prec H(z,t) \qquad (z \in \mathcal{H}) \tag{2.4}$$

where, $g'(z)$ and $g''(z)$ are the first and second derivative of (2.1) defined as follows;

$$g'(z) = 1 + (1+a_2)z + \left(\frac{3}{4} + \frac{3a_2}{2} + \frac{3a_3}{4}\right)z^2 + \left(\frac{1}{2} + \frac{3a_2}{2} + \frac{3a_3}{2} + \frac{a_4}{2}\right)z^3 + \left(\frac{5}{16} + \frac{5a_2}{4} + \frac{15a_3}{8} + \frac{5a_4}{4} + \frac{5a_5}{16}\right)z^4 + \cdots, \tag{2.5}$$

$$g''(z) = (1+a_2) + \left(\frac{3}{2} + 3a_2 + \frac{3a_3}{2}\right)z + \left(\frac{3}{2} + \frac{9a_2}{2} + \frac{7a_3}{2} + \frac{3a_4}{2}\right)z^2 + \left(\frac{5}{4} + 5a_2 + \frac{15a_3}{2} + 5a_4 + \frac{5a_5}{4}\right)z^3 + \cdots. \tag{2.6}$$

3. RESULTS

In this section, motivated by the work of Yasamian et.al (2020) [18] and by the principle of subordination, the initial coefficients for the class $\mathcal{M}_D(\tau, t)$ are obtained. These initial coefficient bounds satisfies and agrees with recent works using the Chebyshev polynomial. Furthermore, the Fekete-Szego which finds application in the theory of singularities were obtained.

Theorem 3.1. *If $g(z)$ given in (2.1) belongs to the class $\mathcal{M}_D(\tau, t) : 0 \leq \tau \leq 1$, then;*

$$|a_2| \leq \frac{4t}{(1+\tau)} + 1,$$

$$|a_3| \leq \frac{8t^2 + 4t + 2\tau + 1}{2(\frac{1}{2} + \tau)} + \frac{(2t\tau)(4\tau + 2)}{(\frac{1}{2} + \tau)(1+\tau)} + \frac{(4t^2)(1+3\tau)}{(\frac{1}{2} + \tau)(1+\tau)^2},$$

and

$$|a_4| \leq \frac{8t(21\tau^2 + 3\tau - 2)}{(1+3\tau)(1+\tau)} + \frac{16t^2}{(1+\tau)^2} - \frac{16t^2 + 12t + 3\tau - 3}{(1+3\tau)} + \frac{2t(1+5\tau) + 4t(1+5\tau)(\tau + 2t + 4t^2)}{(\frac{1}{2} + \tau)(1+3\tau)(1+\tau)} + \frac{16t^2(10\tau^2 + 16\tau + 6)}{(1+3\tau)(1+\tau)^2} + \frac{16t^3(1+5\tau)}{(\frac{1}{2} + \tau)(1+\tau)^3} + \frac{(1+5\tau)(1 - 2(\tau + 2t + 4t^2))}{2(\frac{1}{2} + \tau)(1+3\tau)} + \frac{(1+5\tau)(4t^2)}{(\frac{1}{2} + \tau)(1+\tau)^2} + \frac{(1+7\tau)(4t)^3}{(3+9\tau)(1+\tau)^3} + \frac{64t^3 + 64t^2 - 16t + 7\tau - 15}{(3+9\tau)}.$$

Proof. Assume $g \in \mathcal{M}_D(\tau, t)$, then from (2.4) and (2.8);

$$(1-\tau)\frac{zg'(z)}{g(z)} + \tau\left(1 + \frac{zg''(z)}{g'(z)}\right) = H(\omega(z,t)) \tag{3.1}$$

where,

$$H(\omega(z,t)) = 1 + U_1(t)\omega(z) + U_2(t)\omega^2(z) + U_3(t)\omega^3(z) + \cdots \tag{3.2}$$

Then from (2.2);

$$\omega(z) = e_1 z + e_2 z^2 + e_3 z^3 + e_4 z^4 + \cdots$$

$$\omega^2(z) = e_1^2 z^2 + 2e_1 e_2 z^3 + (2e_1 e_3 + e_2^2)z^4 + (e_1 e_4 + 2e_2 e_3)z^5 + \cdots$$

$$\omega^3(z) = e_1^3 z^3 + 3e_1^2 e_2 z^4 + 3(e_1^2 e_3 + e_1 e_2^2)z^5 + \cdots$$

$$\omega^4(z) = e_1^4 z^4 + 4e_1^3 e_2 z^5 + \cdots$$

$$\omega^5(z) = e_1^5 z^5 + \cdots$$

Thus,

$$H(\omega(z,t)) = 1 + U_1(t)e_1 z + [U_1(t)e_2 + U_2(t)e_1^2]z^2 + [U_1(t)e_3 + 2U_2(t)e_1 e_2 + U_3(t)e_1^3]z^3 + \cdots$$

Therefore,

$$\begin{aligned}&(1-\tau)\frac{zg'(z)}{g(z)}+\tau\left(1+\frac{zg''(z)}{g'(z)}\right)=1+U_1(t)e_1z+\\&[U_1(t)e_2+U_2(t)e_1^2]z^2+\Bigg[U_1(t)e_3+2U_2(t)e_1e_2\\&+U_3(t)e_1^3\Bigg]z^3+\cdots\end{aligned}\tag{3.3}$$

Hence,

$$\begin{aligned}&(1-\tau)\frac{zg'(z)}{g(z)}=\\&\frac{(1-\tau)\left[z+(1+a_2)z^2+\left(\frac{3}{4}+\frac{3a_2}{2}+\frac{3a_3}{4}\right)z^3+\right.}{z+\left(\frac{1}{2}+\frac{a_2}{2}\right)z^2+\left(\frac{1}{4}+\frac{a_2}{2}+\frac{a_3}{4}\right)z^3}\\&\frac{\left.\left(\frac{1}{2}+\frac{3a_2}{2}+\frac{3a_3}{2}+\frac{a_4}{2}\right)z^4+\cdots\right]}{+\left(\frac{1}{8}+\frac{3a_2}{8}+\frac{3a_3}{8}+\frac{a_4}{8}\right)z^4+\cdots}\end{aligned}\tag{3.4}$$

$$\begin{aligned}&\tau\left(1+\frac{zg''(z)}{g'(z)}\right)=\\&=\frac{\tau\left[1+2(1+a_2)z+\left(\frac{9}{4}+\frac{9a_2}{2}+\frac{9a_3}{4}\right)z^2\right.}{1+(1+a_2)z+\left(\frac{3}{4}+\frac{3a_2}{2}+\frac{3a_3}{4}\right)z^2}\\&\frac{+\left(2+6a_2+6a_3+2a_4\right)z^3}{+\left(\frac{1}{2}+\frac{3a_2}{2}+\frac{3a_3}{2}+\frac{a_4}{2}\right)z^3}\\&\frac{\left.+\left(\frac{25}{16}+\frac{25a_2}{4}+\frac{75a_3}{8}+\frac{25a_4}{4}+\frac{25a_5}{16}\right)z^4+\cdots\right]}{+\left(\frac{5}{16}+\frac{5a_2}{4}+\frac{15a_3}{8}+\frac{5a_4}{4}+\frac{5a_5}{16}\right)z^4+\cdots}.\end{aligned}\tag{3.5}$$

Then adding (3.4) and (3.5) with some simple calculation the left hand side of (3.3) and comparing coefficients, we have; For z;

$$(1-\tau)+\tau=1\tag{3.6}$$

For z^2;

$$\Rightarrow\left(1+\tau\right)\left(\frac{1}{2}+\frac{a_2}{2}\right)=U_1(t)e_1\tag{3.7}$$

For z^3;

$$\begin{aligned}&\Rightarrow\left(\frac{1}{2}+a_2+\frac{a_2^2}{2}\right)+\left(\frac{1}{2}+a_2+\frac{a_3}{2}\right)+\tau\left(1+2a_2+a_3\right)\\&-(1+a_2)U_1(t)e_1-\left(\frac{1}{2}+\frac{a_2}{2}\right)U_1(t)e_1\\&=[U_1(t)e_2+U_2(t)e_1^2].\end{aligned}\tag{3.8}$$

Then from (3.7) and (3.8) we obtain;

$$\begin{aligned}&\Rightarrow\left(\frac{1}{2}+\tau\right)a_3+\left(\frac{1}{2}+\frac{1}{2}\tau\right)a_2-\left(\frac{1}{4}+\frac{3}{4}\tau\right)a_2^2\\&+\left(1+\tau\right)\frac{1}{4}=[U_1(t)e_2+U_2(t)e_1^2].\end{aligned}\tag{3.9}$$

For z^4

$$\begin{aligned}&\Rightarrow 7\left(\frac{1}{8}+\frac{3a_2}{8}+\frac{a_2^2}{4}+\frac{a_2a_3}{8}+\frac{a_3}{8}\right)+\left(\frac{3}{8}+\frac{9a_2}{8}+\frac{9a_3}{8}\right.\\&\left.+\frac{3a_4}{8}\right)+9\tau\left(\frac{1}{8}+\frac{3a_2}{8}+\frac{3a_3}{8}+\frac{a_4}{8}\right)+\tau\left(\frac{1}{8}+\frac{3a_2}{8}\right.\\&\left.+\frac{a_2^2}{4}+\frac{a_2a_3}{8}+\frac{a_3}{8}\right)-\left(1+2a_2+a_3\right)U_1(t)e_1-\left(\frac{1}{2}\right.\\&\left.+a_2+\frac{a_2^2}{2}\right)U_1(t)e_1-\left(\frac{3}{2}+\frac{3a_2}{2}\right)[U_1(t)e_2+U_2(t)e_1^2]\\&=[U_1(t)e_3+2U_2(t)e_1e_2+U_3(t)e_1^3].\end{aligned}\tag{3.10}$$

Then from (3.7), (3.9) and (3.10) gives

$$\begin{aligned}&\Rightarrow\left(\frac{3}{8}+\tau\frac{9}{8}\right)a_4+\left(\frac{3}{4}+\tau\frac{3}{2}\right)a_3-\left(\frac{3}{8}+\tau\frac{15}{8}\right)a_2a_3\\&+\left(\frac{1}{8}+\tau\frac{7}{8}\right)a_2^3-\left(\frac{5}{8}+\tau\frac{11}{8}\right)a_2^2+\left(\frac{3}{8}+\tau\frac{3}{8}\right)a_2+\\&\left(1+\tau\right)\frac{1}{8}=[U_1(t)e_3+2U_2(t)e_1e_2+U_3(t)e_1^3]\end{aligned}\tag{3.11}$$

From (3.6) and (3.7) we obtain;

$$\Rightarrow a_2=\frac{4te_1}{(1+\tau)}-1.\tag{3.12}$$

Now we can see from lemma 2.1 that;

$$|a_2|\leq\frac{4t}{(1+\tau)}+1.$$

From (3.9) and (3.12);

$$\begin{aligned}&\Rightarrow a_3=\frac{1}{2(\frac{1}{2}+\tau)}-\frac{2te_1\tau}{(\frac{1}{2}+\tau)(1+\tau)}[4\tau+2]\\&+\frac{4t^2e_1^2}{(\frac{1}{2}+\tau)(1+\tau)^2}[1+3\tau]+\frac{\tau+2te_2+4t^2e_1^2-e_1^2}{(\frac{1}{2}+\tau)}.\end{aligned}\tag{3.13}$$

Then from lemma 2.1 we obtain;

$$\begin{aligned}|a_3|\leq&\frac{8t^2+4t+2\tau+1}{2(\frac{1}{2}+\tau)}+\frac{(2t\tau)(4\tau+2)}{(\frac{1}{2}+\tau)(1+\tau)}\\&+\frac{(4t^2)(1+3\tau)}{(\frac{1}{2}+\tau)(1+\tau)^2}.\end{aligned}$$

From (3.11), (3.12), (3.13) and following the lemma 2.1 we have;

$$
\begin{aligned}
|a_4| \le{}& \frac{8t(21\tau^2+3\tau-2)}{(1+3\tau)(1+\tau)} + \frac{16t^2}{(1+\tau)^2} + \frac{16t^2+12t+3\tau-3}{(1+3\tau)} \\
&+ \frac{2t(1+5\tau)+4t(1+5\tau)(\tau+2t+4t^2)}{(\frac{1}{2}+\tau)(1+3\tau)(1+\tau)} \\
&+ \frac{16t^2(10\tau^2+16\tau+6)}{(1+3\tau)(1+\tau)^2} + \frac{16t^3(1+5\tau)}{(\frac{1}{2}+\tau)(1+\tau)^3} \\
&- \frac{(1+5\tau)(1-2(\tau+2t+4t^2))}{2(\frac{1}{2}+\tau)(1+3\tau)} + \frac{(1+5\tau)(4t^2)}{(\frac{1}{2}+\tau)(1+\tau)^2} \\
&+ \frac{(1+7\tau)(4t)^3}{(3+9\tau)(1+\tau)^3} + \frac{64t^3+64t^2-16t+7\tau-15}{(3+9\tau)}.
\end{aligned}
$$

This completes the proof. □

Fekete-Szego inequalities for the class $\mathcal{M}_D(\tau,t)$

Theorem 3.2. *If $g(z)$ given in (2.1) belongs to the class $\mathcal{M}_D(\tau,t)$ then,*

$$
|a_3-\sigma a_2^2| \le \begin{cases} \frac{2t}{(\frac{1}{2}+\tau)} & \sigma \in [\sigma_1,\sigma_2] \\ \frac{2t}{(\frac{1}{2}+\tau)}\left|\frac{T(\sigma,\tau,t)}{2t(1+\tau)^2} - \frac{(8t^2-2\tau-1)}{4t}\right| & \sigma \notin [\sigma_1,\sigma_2] \end{cases} \tag{3.14}
$$

where

$$
\sigma_1 = \frac{4t(1+\tau)^2+(1+\tau)^2(8t^2-2\tau-1)+8t^2(1+3\tau)}{2(\frac{1}{2}+\tau)(16t^2-8t(1-\tau)+(1+\tau)^2)},
$$

$$
\sigma_2 = \frac{4t(1+\tau)^2+(1+\tau)^2(8t^2-2\tau-1)+8t^2(1+3\tau)}{2(\frac{1}{2}+\tau)(16t^2-8t(1-\tau)+(1+\tau)^2)},
$$

$$
\begin{aligned}
T(\sigma,\tau,t) ={}& \sigma(\frac{1}{2}+\tau)(16t^2-8t(1+\tau)+(1+\tau)^2) \\
&- 4t^2(1+3\tau)+2t\tau(4\tau+2)(1+\tau).
\end{aligned}
$$

Proof. From (3.12) and (3.13)

$$
a_2 = \frac{4te_1}{(1+\tau)} - 1
$$

$$
a_2^2 = \left(\frac{4te_1}{(1+\tau)}\right)^2 - 2\frac{4te_1}{(1+\tau)} + 1
$$

$$
\begin{aligned}
a_3 ={}& \frac{1}{2(\frac{1}{2}+\tau)} - \frac{2te_1\tau(4\tau+2)}{(\frac{1}{2}+\tau)(1+\tau)} + \frac{4t^2e_1^2(1+3\tau)}{(\frac{1}{2}+\tau)(1+\tau)^2} \\
&+ \frac{\tau+2te_2+4t^2e_1^2-e_1^2}{(\frac{1}{2}+\tau)}.
\end{aligned}
$$

then it follows that

$$
\begin{aligned}
|a_3-\sigma a_2^2| \le{}& \frac{2t}{(\frac{1}{2}+\tau)}\Bigg| e_2 - \\
& e_1^2\Bigg[\sigma\frac{(\frac{1}{2}+\tau)(16t^2-8t(1+\tau)+(1+\tau)^2)}{2t(1+\tau)^2} \\
& - \frac{(8t^2-2\tau-1)}{4t} - \frac{2t(1+3\tau)}{(1+\tau)^2} + \frac{\tau(4\tau+2)}{(1+\tau)}\Bigg]\Bigg|.
\end{aligned}
$$

From lemma 2.1, we have

$$
\begin{aligned}
|a_3-\sigma a_2^2| \le{}& \frac{2t}{(\frac{1}{2}+\tau)} \max\Bigg\{1, \\
& \Bigg|\sigma\frac{(\frac{1}{2}+\tau)(16t^2-8t(1+\tau)+(1+\tau)^2)}{2t(1+\tau)^2} \\
& - \frac{(8t^2-2\tau-1)}{4t} - \frac{2t(1+3\tau)}{(1+\tau)^2} + \frac{\tau(4\tau+2)}{(1+\tau)}\Bigg|\Bigg\}.
\end{aligned}
$$

Since $t > 0$, we have

$$
\begin{aligned}
&\Bigg|\sigma\frac{(\frac{1}{2}+\tau)(16t^2-8t(1+\tau)+(1+\tau)^2)}{2t(1+\tau)^2} - \frac{(8t^2-2\tau-1)}{4t} \\
&- \frac{2t(1+3\tau)}{(1+\tau)^2} + \frac{\tau(4\tau+2)}{(1+\tau)}\Bigg| \le 1.
\end{aligned}
$$

$$
\begin{aligned}
&\Leftrightarrow \frac{(8t^2-2\tau-1)}{4t} + \frac{2t(1+3\tau)}{(1+\tau)^2} - \frac{\tau(4\tau+2)}{(1+\tau)} - 1 \\
&\le \sigma\frac{(\frac{1}{2}+\tau)(16t^2-8t(1+\tau)+(1+\tau)^2)}{2t(1+\tau)^2} \le \\
&\frac{(8t^2-2\tau-1)}{4t} + \frac{2t(1+3\tau)}{(1+\tau)^2} - \frac{\tau(4\tau+2)}{(1+\tau)} + 1 \\
&\Leftrightarrow \sigma_1 \le \sigma \le \sigma_2.
\end{aligned}
$$

□

Corollary 3.3. *If $g(z)$ belongs to the class $\mathcal{M}_D(0,t)$ then;*

$$
|a_2| \le 4t+1,
$$

$$
|a_3| \le 16t^2+4t+1
$$

and

$$
|a_4| \le 64t^3+32t^2-\frac{104}{3}t+\frac{7}{3}.
$$

Corollary 3.4. *If $g(z)$ belongs to the class $\mathcal{M}_D(1,t)$ then;*

$$
|a_2| \le 2t+1,
$$

$$
|a_3| \le \frac{16}{3}t^2+\frac{8}{3}t+1
$$

and

$$
|a_4| \le 16t^3+\frac{16}{3}t^2+\frac{52}{3}t+\frac{5}{3}.
$$

Corollary 3.5. *If $g(z)$ belongs to the class $\mathcal{M}_D(0,t)$ then;*

$$
|a_3-\sigma a_2^2| \le \begin{cases} t & \sigma \in [\sigma_1,\sigma_2] \\ \left|\frac{\sigma(16t^2-8t+1)-(8t^2-1)-8t^2}{4}\right| & \sigma \notin [\sigma_1,\sigma_2] \end{cases}
$$

Corollary 3.6. *If $g(z)$ belongs to the class $\mathcal{M}_D(1,t)$ then;*

$$
|a_3-\sigma a_2^2| \le \begin{cases} \frac{4t}{3} & \sigma \in [\sigma_1,\sigma_2] \\ \left|\frac{G(\sigma,t)}{3}\right| & \sigma \notin [\sigma_1,\sigma_2] \end{cases}
$$

where

$$
G(\sigma,t) = \sigma(1+2\tau)(4t^2-4t+1)-(8t^2-3)-8t^2+12t.
$$

4. DISCUSSION

In this work, a novel class of normalized analytic univalent functions, denoted as $\mathcal{M}_D(\tau, t)$, is introduced. These functions are defined through the subordination of Chebyshev polynomials of the second kind. The initial coefficients for this class are derived. Additionally, the study explores the Fekete-Szego functional associated with this class, although the abstract cuts off before detailing the specific findings in this regard. Overall, the work presents a fresh perspective on univalent functions and their connections to Chebyshev polynomials, laying the groundwork for further investigation into their properties and applications for more details [19]–[22].

5. CONCLUSION

In conclusion, this work introduces the class $\mathcal{M}_D(\tau, t)$ of normalized analytic univalent functions, defined by the subordination of the second kind Chebyshev polynomial. Initial coefficients for this class were derived, and the Fekete-Szego functional associated with it was investigated, for more details see [23]–[27].

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