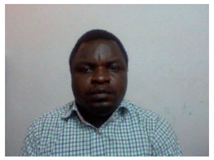
This book is a work for a PhD Mathematics, from Mathematics Department, Modibbo Adama University of Technology, Yola, Nigeria. It deals with the optimal control of a one-machine multiple-product manufacturing system with setup changes, operating in a continuous time dynamic environment. The system is deterministic. When production is switched from one product to the other, a known constant setup time and a setup cost are incurred. Each product has specified constant processing time and constant demand rate, as well as an infinite supply of raw material. The problem was formulated as a feedback control problem. The optimal solution provides the optimal production rate and setup switching epochs as a function of the state of the system (backlog and inventory levels). For the steady state, the optimal cyclic schedule was determined. To solve the transient case, the system's state space was partitioned into [n(n-1)/2], where n is the number of products, mutually exclusive regions such that with each region, the optimal control policy was determined analytically. From the result of analysis, a number of useful recommendations were made. The book is relevant in manufacturing Industries



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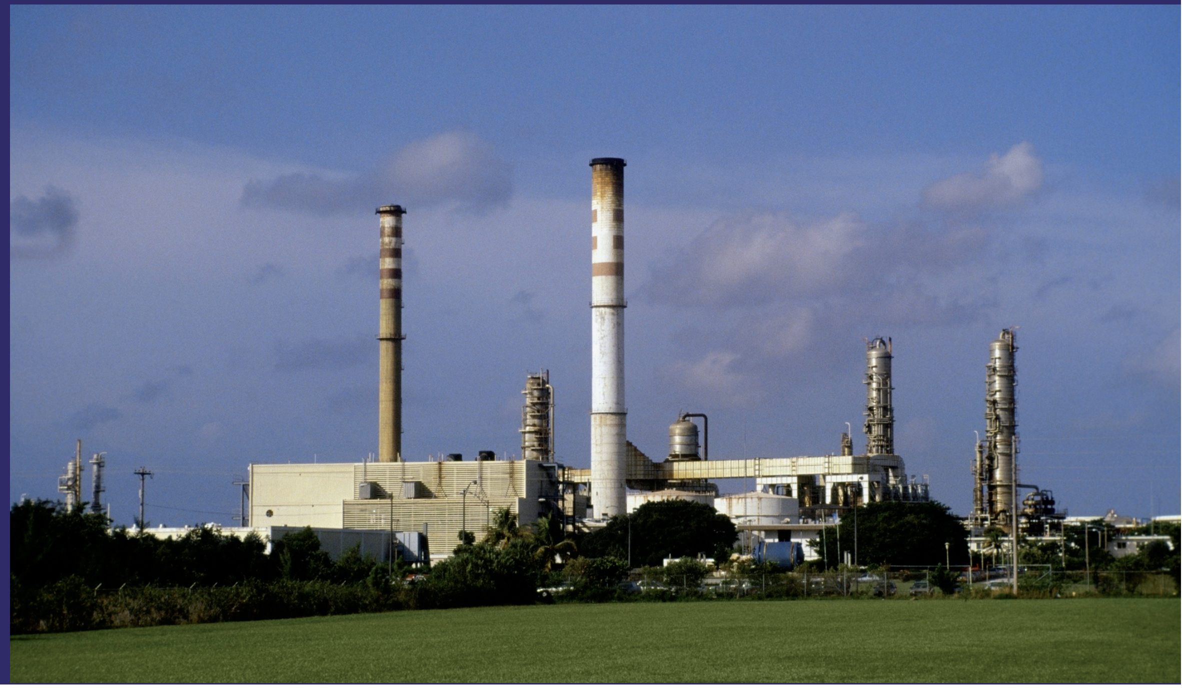
A.S. Optimal Production Control

Onanaye



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DEDICATION

I humbly dedicate this Project to:

i) God Almighty;

ii) My wife, Helen Funmilayo;

iii) My son, Adeoluwa Abraham Kelechi and daughter, Mosopeoluwa Adeola Chidinma, to you all the future belongs.

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TABLE OF CONTENTS

ABSTRACT

This thesis deals with the optimal control of a one-machine multiple-product manufacturing system with setup changes, operating in a continuous time dynamic environment. The system is deterministic. When production is switched from one product to the other, a known constant setup time and a setup cost are incurred. Each product has specified constant processing time and constant demand rate, as well as an infinite supply of raw material. The problem was formulated as a feedback control problem. The optimal solution provides the optimal production rate and setup switching epochs as a function of the state of the system (backlog and inventory levels). For the steady state, the optimal cyclic schedule was determined. To solve the transient case, the system's state space was partitioned into $[n(n-1)/2]$, where n is the number of products, mutually exclusive regions such that with each region, the optimal control policy was determined analytically. From the result of analysis, a number of recommendations were made, which if fully implemented, the consequence is an efficient and effective management of the system leading to optimal productivity.

CHAPTER ONE
GENERAL INTRODUCTION

1.1 Study Background

The idea of a machine that automatically carries out pre-assigned task is now common in today's manufacturing society. The means by which such system is being controlled by the laws that govern their behaviour must be looked into with greater care.

These laws sometimes control variables whose values can be changed by someone acting outside and independent of the system itself. There are mainly two kinds of variables – state variables that define precisely what the system is doing at a time t, and the control variables that can be used to modify the subsequent behaviour of the system (Bai and Elhafsi, 1996).

It is in our interest to consider a system whose behaviour can be described by a set of ordinary differential equations. Such systems are controllable from a given initial state to the desired final state. There is a need for measure of the cost incurred during the transfer at a particular time set up. This is called Time Optimal Control Problem, (Caramanis, Sharifnia, Hu, and Gershwin,, 1991).

Control problems always have constraints on the control variables. That is, minimization of integrals in the presence of constraints (isoperimetric problem) (Enid, 1993).

Consideration shall be on systems whose behaviour can be modelled by a set of *n* ordinary differential equations:

$$\dot{X}_i = f_i(x,u),\ i = 1,2,...,n \tag{1.1}$$

Or in vector form

$$\dot{X}_i = F(x,u) \tag{1.2}$$

These are state equations. The functions f_i are assumed to be defined, continuous and continuously differentiable with respect to $x_1, x_2, ..., x_n$ for all n $\in$ Z^+ and all admissible controls *u*. Assume the system is to be controlled from x_0 at $t = t_0$ to x_1 at $t = t_1$ in such a way that the cost functional

$$J = \int_{t_0}^{t_1} f_0(x,u)dt \tag{1.3}$$

is minimized. Here, x_0, x_1 and t_o are fixed but t_1 is unspecified and assuming also that there are admissible controls that transfer the system from x_0 to x_1 and looking amongst this subject of admissible controls for a control that minimises J.

This kind of control is called an optimal control (Elhafsi and Bai, 1995; 1996a; and Xiao-Lan, 2006).

The aim of the Production and Setup Scheduling Problem (PSSP) (Elhafsi and Bai, 1996a) is to determine the optimal production rates and setup epochs of several products on a single machine. The system is believed to have controllable production rates. There is known constant demand rate and processing time for each product. Any time there is a production switch from one product to the next, a constraint setup time as well as a fixed setup cost are incurred. Backlog is allowed and the system is not necessarily in steady state. The objective here is to control the rate of production of each product as well as to control the setup epochs so as to minimise the total setup, inventory and backlog costs over a finite or infinite planning horizon.

The problem therefore, reduces to the Economic Lot Scheduling Problem (ELSP) (Elhafsi and Bai, 1996b). When the planning horizon is finite, the system is in steady state, the machine has fixed production rates, no backlog is allowed, and the objective is to determine lot sizes that minimize the average setup and inventory holding costs per unit time. The PSSP is focused on the state of the system during its transient and steady state periods, spanning through the planning horizon (schedule). In this work, we are considering a finite planning schedule and assume that the steady state is finite time achievable. This model is believed to have many applications in the real world problems.

A case study of a bottleneck machine is to be used for the production planning of several products, periodically (say quarterly). The demand rate of each period of time is forecasted at the beginning of that period and should be satisfied during the planning period. The system is believed also to have enough capacity to meet the forecasted demand for each product and is fast enough to bring the inventory/backlog of all products to a steady state (which constitutes the most economical way of operating the system). At the end of the planning schedule, the inventory and backlog of the products are checked and possible different forecasted demand rates are to be used in the next planning period. It is clear that the output (inventory/backlog) of the previous period constitute the input for the current period and hence new initial conditions as well as new parameters are to be used. The PSSP is designed as a feedback control problem.

The control must respond to certain initial disruptions so as to minimize a certain criterion. In this work, the study focused on the production and setup control of a deterministic one-machine multi-type system within a feedback control framework. The work is based on the previous of Elhafsi and Bai, (1996a); and Onanaye, (2006), on the optimal production control of a dynamic two-product, and three-product manufacturing system with setup costs and setup time respectively. The steady state solutions of optimal control of a one-machine three-product, four-product and five-product manufacturing systems with setup changes, operating in a continuous time dynamic environment had been considered by Onanaye and Odekunle, (2009a); Onanaye and Odekunle, (2009b); and Onanaye, Odekunle, and Egwurube, (2011).

1.2 Problem Formulation and Design

Consider a one-machine manufacturing system producing multi-distinct products, each has a constant demand rate $d_i(i=1,2,3,...,n)$ when production is switched from product Type j to product Type $i(j \neq i \neq l \neq ... \neq z)$, given that constant time δ_i and setup cost $k_i(i=1,2,3,...,n)$ are incurred.

This formulation follows the general framework introduced by Kimemia and Gershwin (1983), where the production flows is modelled as continuous rather than discrete. Let $x_i(t)$ be the production surplus of product Type $i(i=1,2,3,...,n)$ at time t; a positive value of $x_i(t)$ represents inventory while a negative value represents backlog.

Let $u_i(t)$ be the controlled production rate of machine producing Type i product at time t.

Let $\sigma(t)=\{\sigma_{100...0,}(t),\sigma_{0200...0}(t),...,\sigma_{000...z}(t),...\}$

be the set of state vectors of the machine at time t. Then,

$\sigma_i(t),\sigma_{ijlm}(t),\quad [\mathrm{i} \neq \mathrm{j} \neq 1 \neq m \neq ... \neq z,\ \mathrm{i}=1,2,3,...\mathrm{n}\ \ \mathrm{j}=1,2,3,...\mathrm{n}\ \ 1=1,2,3,...\mathrm{n},\ \mathrm{m}=1,2,3,...,n\ ....]$

are right continuous binary functions of t, such that

$\sigma_i(t)=1$ when the machine is ready to produce Type i product, otherwise $\sigma_i(t)=0$ and also when the machine is undergoing a setup change from Type j to product Type i and from Type i to Type j to Type l, …, and from Type y to Type z, $\sigma_{ijlm...z}(t) = 0$

Let $s(t)$ be a non negative right continuous function of t which takes on the value δ_i at the beginning of each setup change to Type $i(i = 1,2,3,...,n)$ and decreases with time. $s(t)$ indicates whether a setup is completed or not.

It is assumed that initially, the machine is not setup for either product Type.

1.2.1 System Dynamic and Constraint

The dynamics of the system can be described by:

$$\frac{dx_i(t)}{dt} = u_i - d_i \quad i(i = 1,2,3,...,n) \tag{1.4}$$

$$0 \leq u_1(t) \leq U_i \, \sigma_i(t), \qquad i(i = 1,2,3,...,n) \tag{1.5}$$

where U_i is the maximum production rate of the machine when it is producing Type i product.

When stable, the setup states of the machine obey the following set of constraints.

$$\sigma_1(t) + \sigma_2(t) + \sigma_3(t) + \sigma_4(t) + \sigma_5(t) + ... + \sigma_z(t) + ... = 1$$

otherwise is zero.

$$\textit{If } \sigma_i(t^-) = 1 \text{ and } \sigma_i(t^+) = 0 \text{ then } s(t) = \delta_j \text{ and } \sigma_{ijlm...z}(t) = 1 \tag{1.7}$$

$$\textit{If } s(t^-) > 0 \text{ and } \sigma_{ijlm...z}(t^-) = 1 \text{ then } \dot{s}(t) = -1 \text{ and } \sigma_{ijlm...z}(t) = 1 \tag{1.8}$$

$$\textit{If } s(t^-) = 0 \text{ and } \sigma_{ijlm...z}(t^-) = 1 \text{ then } \sigma_{ijlm...z}(t) = 0 \text{ and } s(t) = 0 \text{ and } \sigma_j(t) = 1 \tag{1.9}$$

For $i = 1,2,3,...,n$; $j = 1,2,3,...,n$;, $m = 1,2,3,...,n$; ,…,; $z = 1,2,3,...,n$, $i \neq j \neq 1 \neq m \neq ... \neq z$

Where $\dot{s}(t)$ denote the time derivative of $s(t)$.

1.2.2 Penalty Function

For mathematical convenience, it is assumed that setup costs are incurred at a constant rate

$$X_i = k_i / d_i \qquad (i = 1,2,3,...,n) \tag{1.10}$$

Naira per unit time, during a setup change. Hence, at the end of a setup change of product Type i, having a total cost of k_i, the instantaneous cost which penalises the production for being ahead of (i.e. $x_i > 0$) or being behind of (i.e. $x_i < 0$) the demand is given by:

$$h(x) = \sum_{i=1}^{i=n} [c_i^+ x_i^+(t) + c_i^- x_i^-(t)] \tag{1.11}$$

where c_i^+ *and* c_i^- are the per unit instantaneous inventory holding and backlog costs respectively, and $x_i^+(t) =$ max {x_i(t), 0}; $x_i^-(t) =$ max {-x_i(t),0}.

The total instantaneous cost is given by:

$$\begin{aligned} g(x,\sigma) &= h(x) + \sum_{i=1,\, j\neq i\neq l\neq \ldots\neq z}^{i=n} x_i \sigma_{ijkl\ldots z}(t) \\ &= \sum_{i=1,\, j\neq i\neq l\neq \ldots\neq z}^{i=n} [c_i^+ \left|x_i^+(t)\right| - c_i^- \mid x_i^-(t) \mid + (k_i / \delta_i) \sigma_{ijkl\ldots z}(t)] \end{aligned} \tag{1.12}$$

1.2.3 State Variables and Control Variables

The state variable of the system is given by the vector

$$X(t) = [x_1(t), x_2(t), x_3(t), x_4(t), \ldots, x_n(t)] \tag{1.13}$$

The variables:

$$U(t) = [u_1(t), u_2(t), u_3(t), u_4(t), \ldots, u_n(t)] \tag{1.14}$$

and

$$\sigma(t) = \{\sigma_{100\ldots0}(t), \sigma_{0200\ldots0}(t), \ldots, \sigma_{000\ldots z}(t), \ldots\} \tag{1.15}$$

are the control variables; and denote by (σ, u) the compLete control vector.

1.2.4 Capacity Set

The capacity set represents the set of feasible production rates.

When the set up state is σ(t) at the time *t,* it is given by:

$$\Omega[\sigma(t)] = \{u(t) \mid 0 \le u_i(t) \le U_i \sigma_i(t),\ \ i = 1,2,3,\ldots n\} \tag{1.16}$$

1.2.5 Setup Constraints Set

The setup constraints set is the set of all possible setup vectors

$$\sigma(t) = \{\sigma_{100\ldots0}(t), \sigma_{0200\ldots0}(t), \ldots, \sigma_{000\ldots z}(t), \ldots\}$$

satisfying constraints (1.5) to (1.9)

Let ϕ denote this set.

1.2.6 Admissible Control Policies

Denote by $\psi(\Omega,\ \phi)$ the set of feasible controls, which depends on ϕ and Ω such that

$$\mu : Z^n \to \psi(\Omega,\ \phi)$$

and

$$\mu(x) = (\sigma, u)$$

are piecewise continuously differentiable. These admissible control policies are feedback controls that specify the control actions (setup and production rate of the machine) to be taken, given the state of the system.

1.2.7 Objective Function

The objective is to determine an optional control policy for $\mu^* \in \beta$

Corresponding to a setup control

$U(t) = [u_1(t), u_2(t), u_3(t), u_4(t), ..., u_n(t)]$

$$\sigma^* = \{\sigma_{100...0}{}^*, \sigma_{0200...0}{}^*, ..., \sigma_{000...z}{}^*, ...\} \tag{1.17}$$

and a production rate control

$$U^* = [u^*{}_1, u^*{}_2, u^*{}_3, u^*{}_4, ..., u^*{}_n] \tag{1.18}$$

that minimises for each initial state $x_i(t)$ the following cost function

$$J_\mu(x(t), t) = \int_t^{t_f} g(x(s), \sigma(s))ds \tag{1.19}$$

where the minimization is over all functions

$$\mu[x(t)] = [\sigma(\tau), u(\tau)] \tag{1.20}$$

such that

$x(\tau), \sigma(\tau)$ *and* $\mu(\tau)$ *satisfy* (1.5) *and* $\sigma(\tau)$, $\mu(\tau) \in \psi(\Omega, \theta)$

For $t \le \tau \le t_f$, where t is assumed to be sufficiently large.

1.3 Aim and Objectives of the study

The aim is to get optimal solution of the production and set up scheduling problem of a dynamic n-product manufacturing system that minimises incurred costs of production.

To realise the above aim, the following objectives would be achieved at the end of the study:

(i) To know the nature of optimal control problems.

(ii) To minimise the total backlog, inventory and setup costs incurred over a finite horizon in a dynamic multiple-product manufacturing system.

(iii) To determine the optimal production rate and set up switching epochs as a function the state of the system (backlog/inventory levels).

(iv) To determine the optimal cyclic schedule for the steady state.

(v) To solve the transient case in the system

(vi) To determine analytically the optimal control policy.

(vii) To recommend based on the research findings, areas of application of the study in today's manufacturing society and further area of research if there is any.

1.4 Significance of the Study

The significance of the study is in the application of Applied Mathematics (optimization techniques) in solving related problems, being seen as an essential tool for today's production/operation managers (engineers) in manufacturing society. This study is relevant to companies which produce more than one product yet from the same machine.

The examples of such companies include Cadbury Nigeria PLC, Nigerian Bottling Company (Coca-Cola Products), Nigerian Breweries PLC and others.

1.5 Justification of the Study

The justification for this study is the fact that we are in a world of scarce resources (resources such as time, human, land, capital and so on) and restrained economy where by adequate and optimal utilisation of these scarce resources can not be over emphasised.

Therefore, any manufacturing system that fails to utilise optimization tools, such as PSSP, to minimise costs of production that will maximise returns on investments, will in no distance time be out of business in the present global competitive economy.

1.6 Definition of Basic Terms

1. Deterministic System: A system in which the later states of the system follows, or are determined by, the earlier ones.

2. Controller: In control theory, a controller is a device, possibly in the form of a chip, analogue electronics, or computer, which monitors and physically alters the operating conditions of a given dynamical system.

3. Dynamical System: It is a concept in Mathematics where a fixed rule describes the time dependence of a point in a geometrical space. Dynamical system has a state given by a set of real numbers (a vector) that can be represented by a point in an appropriate state space (a geometrical manifold).

4. Cost Function: Cost function is a function of input prices and output quantity. Its value is the cost of making that output given those input prices.

A common form:

$c(w_1, w_2, y)$ is the cost of making output quantity y using inputs w_1 *and* w_2 per unit. It is a mathematical formula used to predict the cost associated with a certain action or a certain level of output.

5. Cost Functional: It is a mathematical expression giving the processing (travelling) time as a function of the production rate (speed), geometrical considerations, and initial conditions of the system. (See section 2.2 for more explanation).

6. Asymptotically Optimal: In Computer Science, an algorithm is asymptotically optimal with respect to a particular resource if the problem has been proven to require $\Omega(f(n))$ of that resource, and the algorithm has been proven to use only $O(f(n))$.

(All the above definitions were sourced from Wikipedia, the free internet based encyclopaedia). Other form of definitions of terms can be found in the Chapter Two of the thesis.

CHAPTER TWO
LITERATURE REVIEW

2.1 Optimisation Theory and Optimal Control Concept

The shortest distance between two points in a plane is obtained by drawing the straight line that joins them. The shape that encloses maximum area of a given length of perimeter is called circle. These well-known facts were known to the ancient Greeks and are oldest known solutions in the class of problems discussed in what is now called optimization theory. The theory of optimal control is a branch in the studies of optimization and was developed in the 1950s in response to American and Russian efforts to explore the solar system.

Enid (1993) believed that modern optimal control can be looked at as a development of the classical theory of Euler and Lagrange – a study of admissible functions that are less well-behaved.

2.2 Optimal Control Problem Defined

According to Enid (1993), optimal control problem can be defined as follows: Let the state of the system at time t be a vector ***X(t)*** in an n-dimensional Euclidean space which can be called the state space X. As the system evolves in time, it is expected of ***X(t)*** to trace out a continuous path in its space.

The steering device or control is modeled as an r-dimensional vector function of time u.

The components of u are allowed to be piecewise continuous and the values they can take are bounded so that any time t, u lies in some bounded region u of the control space. For instance if r =2, so that $u = (u_1, u_2)$ and by imposition of the restriction $|u_i| \leq 1, i = 1, 2$ then u is the unit square in the plane.

Such controls are deemed admissible. He studied systems whose behaviour can be modelled by a set of n ordinary differential equations:

$$\dot{X}_i = f_i(x, u) \; i = 1, 2, 3, \ldots, n \tag{2.21}$$

Or in vector form

$$\dot{X}_i = f(x, u) \tag{2.22}$$

He called these, the state equations. The functions f_i are assumed to be defined, continuous and continuously differentiable with respect to $x_1, x_2, \ldots, x_n$ for all $x_i \in X$ and all admissible controls u. According to him, these assumptions ensure that, given an initial condition $X(t_0) = X^0$, there is a unique continuous solution for the state equations satisfying $X = X^0$ at $t = t_0$. He noted that the solution is continuous even though the control is piecewise continuous and that the system is controlled from x^0 at t_0 to x^1 at $t = t_1$ in such a way that the cost functional

$$J = \int_{t_0}^{t_1} f_0(x,u)dt \tag{2.23}$$

is minimized.

Here, x^0, x^1 and t_0 are fixed but t_1 is unspecified. Such a control will be called an Optimal Control. Burley (1974), described a typical example of an optimal control mechanism as a rider on a bicycle. The rider travels with a constant speed in a straight line path and for some reason he deviates from this path. He then adjusts the handle bars of the cycle to bring himself back to his steady path as quickly as possible. From the mechanics of the bicycle, there is a complicated set of differential equations governing its motion, these incorporate an external or control force provided by the rider on the front wheel via the handle bars. According to him, taught by a little experience, the human brain easily provides the solution to this very complicated optimal control problem. However, if the controlling is to be done automatically by a computer, there is the solution of the governing differential equations to be obtained and an optimization routine to be followed. To identify the difficulty, he considered the problem of finding the function $u(t)$ which minimizes the cost functional

$$I[u] = \int_0^T f(t,x,\dot{x},u)dt \tag{2.24}$$

where

$$\dot{x} = g(t,x,u),\ x(0) = 1. \tag{2.25}$$

For each u(t) the differential equation (2.25) can be solved (at least in principle), subject to the given boundary condition, to give the function $x(t)$. These x(t), u(t), can then be substituted back into (2.24) to give the cost. This method can be followed for each u(t), the costs compared and the minimum chosen.

2.3 Review of Models of Manufacturing Systems

Elhafsi and Bai (1996b) reviewed that despite a large amount of research work on the Economic Lot Scheduling Problem (ELSP) an optimal solution approach has not been proposed yet. Rather, good (sometimes excellent) heuristics approaches have been suggested. According to them, comprehensive review of the ELSP through 1976 is given in Elmaghraby (1978).

The Production and Setup Scheduling Problem (PSSP) is designed as a feedback control problem. The control here must respond to certain direction,. Caramanis, Sharifnia, Hu and Gershwin (1991) derived an Optimality Conditions for setup changes and solved them numerically for a two-part Type system using a quadratic cost criterion. Hu and Caramanis (1992) solved the three-part Type setup problem numerically and deduced structural properties of the optimal policies. Based on the numerical results, they proposed near-optimal policies. Bai and Elhafsi (1993) studied the real-time scheduling of an unreliable one-machine two-part-Type non-resumable setup system. They provided a continuous dynamic programming formulation of the problem which they discretised and solved numerically.

Based on the numerical solution, they provided two heuristics to solve the stochastic problem. Akellar and Kumar (1986) deal with a single machine (with two states: up and down,), single product problem. They obtained an explicit solution for the threshold inventory level, in terms of which the optimal policy is as follows: whenever the machine is up, produce at the minimum rate if the inventory level is less than the threshold, produce on demand if the inventory level is exactly equal to the threshold, and not produce at all if the inventory level exceeds the threshold. Because of the simple structure of their problem, they were able to obtain an explicit solution of the problem, and thus verify the optimality of the resulting policy. They showed that the so-called hedging point policy is optimal in their simple model. It is a policy to produce at full capacity if the surplus is smaller than a threshold level, and produce anything if the surplus is higher than the threshold level, and produce as much as possible but not more than the demand rate when the surplus is at the threshold level. Srivastan and Dallery (1998) generalised the Bielecki-Kumar problem to allow for two products. They limited their focus to only the class of hedging point policies and to partially characterise an optimal solution within that class.

Feng and Yan (1999) extend the Akella-Kumar model to the case where the exogenous demand from a homogenous poison flow. They proved the optimality of the threshold control Type for this extended Akella-Kumar model.

When their problem was generalised to convex costs and more than two machine states, it was no longer possible to obtain explicit solutions. Using viscosity solution technique, Sethi, Soner, Zhang and Jiang (1992) investigate this general problem.

They studied the elementary properties of the value function. They showed that the value function is a convex function and that it is strictly convex, provided the inventory cost is strictly convex. Moreover, it was shown to be a viscosity solution to the Homilton-Jacobi-Bellman (HJB) equation and to have upper and lower bounds each with polynomial growth.

Following the notation given by Sethi et al. (1992), Let x(t), u(t), z, and m(t) denote respectively, the inventory level, the production rate, the demand rate, and the machine capacity level at time t, $t \in [0,\infty]$

They assumed that $x(t) \in R^n, u(t) \in R_+^n, Z$ is a constant positive vector in R_+^n.

Furthermore, they assumed that m($\cdot$) is a Markov process with a finite space

M = {0, 1,..., p}

They wrote the dynamics of the system as

$$x(t) = u(t) - z, x(0) = x^0 \qquad (2.31)$$

They put forward the following definitions:

Definition 1: A control process (production rate $u(\cdot) = \{u(t) : t \geq 0\}$) is called admissible with respect to the initial capacity m if

(i) $u(\cdot)$ is adapted to the filtration $\{F_t : t \geq 0\}$) with $F_t = \sigma\{m(s) : 0 \leq s \leq t\}$, *the* $\sigma-$ field generated by m(t)

(ii) $0 \leq r.u(t) \leq m(t)$ for all $t \geq 0$ for the same positive vector r.

Let $A(m)$ denote the set of all admissible control with the initial condition $m(0) = m$.

Definition 2: A real-valued function $U(x, m)$ on $R^n \times M$ is called an admissible feedback control, or simply feedback control, if

i) for any given initial x, the equation

$$x(t) = u(x(t), m(t) - z), \quad x(0) = x, \qquad (2.32)$$

has a unique solution ;

ii) $$u(\cdot) = \{u(t) = u[x(t), m(t)] : t \geq 0\} \in A(m) \qquad (2.33)$$

Let h(x) and c(u) denote the surplus cost and production cost function, respectively.

For every u($\cdot$) $\in$ A(m), $x(0) = x$ and m(0), Let the cost function of the system be defined as follows:

$$J[x,m,u(\cdot)] = = E\int_0^\infty e^{-\ell t}[h(x(t)) + c(u(t))]dt \qquad (2.34)$$

where $\ell > 0$ is the given discount rate.

The problem is to choose an admissible control $u(\cdot)$ that minimises the cost function $J(x,m,u(\cdot))$. They defined the value function as;

$V(x,m) = \inf u(\cdot) \in A(m) J(x,m,u(\cdot))$ (2.35)

We now make the following assumptions on the cost functions *h(x)* and *c(u)*.

Assumption 1: If *h(x)* is a nonnegative convex function with *h*(0) = 0, there are positive constants $c_{21}, c_{22}, c_{23}, k_{12} \geq 0, k_{22} \geq 0$

such that

$c_{21} |x|^{k_{21}} - c_{22} \leq h(x) \leq c_{23}(1 + |x|^{k_{22}})$.

Assumption 2: *c*(*u*) is a nonnegative function, *c*(0) = 0, and *c*(*u*) is twice differentiable. Moreover, *c*(*u*) is either strictly convex or linear.

Assumption 3: *m*(·) is a finite state Markov chain with generator Q, where

Q = $q_{ij;}$ $i, j \in m$

is a (p +1) * (p+1) matrix such that $q_{ij} \geq 0$ for $i \neq j$

and

$$q_{ij} = -\sum\nolimits_{i \neq j} q_{ij}. \quad (2.36)$$

That is, for any function f(·) on *M*

$$Qf(\cdot)(m) = \sum_{i=m} q_{mi} [f(i) - f(m)]. \quad (2.37)$$

With these three assumptions, we state the following theorem concerned with the property of the value function $V(\cdot,\cdot)$, and proved in Fleming, Sethi and Soner (1987).

Theorem 2.3.1:

(i) For each $m, v(\cdot, m)$ is convex on R^n and $v(\cdot, m)$ is strictly convex if $h(\cdot)$ is so.

(ii) There exist positive constants c_{24}, c_{25} *and* c_{26} such that for each $mc_{24} |x|^{k_{21}} - c_{25} \leq v(x,m) \leq c_{26}(1 + |x|^{k_{22}})$, where k_{21} and k_{22} are the power indices in assumption 1.

The rest of this can be found in their work.

Gershwin and associates (Connoly, Dallery and Gershwin (1992), Van, Lou and Gershwin (1993), Viovelle (1993) and Viovelle and Gershwin (1991)) carried out a good deal of computation work in connection with manufacturing systems. Their works include style of parallel machine systems described earlier in the review work as well as no-wait flow shops (or flow shops without internal buffers (treated in Kimemia and Gershwin (1983)).

Darakananda (1989) developed a simulation software called Hiercsim based on the control algorithms of Gershwin, et'al (1985) and Gershwin (1989).

It should be noted that controls constructed in these algorithms have been shown under some conditions to be asymptotically optimal by Sethi and Zhang (1994b) and Sethi, Zhang and Zhou (1994).

One of the main weakness of the early version of Hiercsim for the purpose of this review was its inability to deal with the internal storage, see also VioLette and Gershwin (1991). Bai (1991) and Bai and Gershwin (1990) developed a hierarchical scheme based on partitioning machines in the original flowshops or job shop into a number of virtual machines each devoted to a single part Type production. VioLette (1993) developed a modified version of Hiercsim to incorporate the method of Bai and Gershwin (1990). VioLette and Gershwin (1991) performed a simulation study indicating that the modified method is efficient and effective.

As stated earlier, Sethi and Zhou (1996b) constructed asymptotically optimal hierarchal controls $u^{\varepsilon}(x,m)$. Samaratunga, Sethi and Zhou (1997) compared the performance of their hierarchal controls (HC) to that of optimal control (OC) and two other existing heuristic methods known as Kanban Control (KC) and two boundary control (TBC). Like HC, KC is a two parameter policy defined as follows:

$$U^{\varepsilon}kc(x,m)=\begin{cases}(m,0)\,if\;0\le x_1\le\theta_1(\varepsilon)>\theta_2(\varepsilon)\\ u^{\varepsilon}(x,m) & ,otherwise\end{cases} \tag{2.38}$$

Note that KC is a threshold – Type policy.

TBC is a three – parameter policy developed by Lou and Van Ryzin (1989). Pressman, Sethi, Zhang and Zhang (1989a) extended the hedging point policy to the problem of Two part types by using the potential function related to the dynamic programming equation of the two part problem. Using a verification theorem for the two part Type problem, they proved the optimality of the hedging point policy.

For the deterministic model, Sethi, Zhang and Zhang (1996) provide optimal control policies and the optimal value. The difficulty in proving the optimality of these cases rather than special cases lies in the fact that when the problem is generalised to include convex costs and multiple machine capacity levels, explicit solution are no longer possible. The only way out is to develop appropriate dynamic programming equations and then investigate the existence of their solutions, and necessary theorems for optimality.

Ciufudean, et'al (2009) considered a discrete manufacturing system whose production rates are stochastic and not equal. They focused on transient analysis of Markov models. They were able to decompose the system which enabled them to determine the event occurrence such as: a machine fails, a machine is repaired, a buffer fills up and a buffer empties. The global properties of machines were determined with the computation in Markov models. At the end they were able to show that a discrete-event system formulation and state partition into basic cells and fast and slow varying section, lead to a reduced computation cost.

Onanaye and Odekunle (2009a) worked on steady state solution of optimal production control of a dynamic three-product manufacturing system with setup cost and setup time. The work was based on previous work of Onanaye (2006). Onanaye and Odekunle (2009b) expanded on the previous work to four-product manufacturing system. Onanaye, Odekunle and Egwurube (2011) did more work on the results of Onanaye and Odekunle (2009b) by extending it to a five-product manufacturing system, they were able to get optimal scheduling cycle for the system considered.

2.4 Remarks

In this chapter, effort was made in reviewing the concept of optimisation, optimal control and various models of manufacturing system consisting of compLetely flexible machines for which a theory of optimal control has been developed and for which hierarchal controls that are asymptotically optimal have been constructed. Attempt was made to look at a system in which production capacity and/or demand are random, in which state constraints may or may not be present, and in which there are flow and multiple hierarchical decision levels. Different criteria were considered as well. These are discounted costs, long-run average costs, risk-sensitive controls with discounted costs and risk-sensitive controls with average costs.

CHAPTER THREE

METHODOLOGY

3.1 Preliminary

It can be shown (see Bai and Elhafsi, 1996; and Onanaye, 2006) that the optimal solution to the problem formulated in section 1.2 can be obtained in two parts consisting of a transient period and a steady state period. The steady state corresponds to the case where the state of the system (inventory/backlog) has already reached a cyclic schedule, and where the produced lots of each product Type are of constant size over time.

On the other hand, the transient period corresponds to the case where the state of the system still has not reached the cyclic schedule. In this section, a theorem that will allow the reduction of the set of feasible production rates from an infinite set to a finite one with only three possible production rates will be stated and proved.

Let t_s be the time instant the system reaches the steady state.

The total cost can then be written as;

$$J_{\mu}(x(t)t) = \int_{t}^{t_s} g(k(s), \sigma(s))ds + \int_{t_s}^{t_f} g(x(s), \sigma(s))ds \tag{3.1}$$

$$= J_{\mu}^{T}[(x(t), t) + (t_f - t_s)]_{\mu}^{T}(x(t_s), (t_s)$$

where $J_{\mu}^{T}(x(t), t)$ is referred to as the transient cost

$$J_{\mu}^{T}(x(t), t) = \int_{t}^{t_f} g(x(s), \sigma(s))ds \tag{3.2}$$

and $J_{\mu}^{T}(x(t_s), t_s)$ is the average steady state cost,

$$J_{\mu}^{T}(x(t), t) = (t_f - t_s)^{-1} \int_{t}^{t_f} g(x(s), \sigma(s))ds \tag{3.3}$$

Throughout this work, it shall be assumed that the following conditions hold.

Assumption 1: It is assumed that (t_f – t), the planning horizon is long enough so that the system reaches the steady state and stays there for a long period. The following Lemma is based on Assumption 1.

Lemma 1: Let t_s be the time the system reaches the steady state, and Let T be the length of the cyclic schedule. Then, it follows that

$$\lim \tfrac{1}{(t_f - t_s)} \int_t^{t_f} g(x(s), \sigma(s))ds = \frac{1}{T} \int_t^{t_f} g(x(s), \sigma(s))ds \qquad (3.4)$$

$t_{f \to T}$

$t_s \to 0$

and from Assumption 1, it follows that

$$J_\mu^s(x(t_s), t_s) = \tfrac{1}{T} \int_t^{t_f} g(x(s), \sigma(s))ds \qquad (3.5)$$

[Source: Modification of Lemma by Elhafsi and Bai (1996b)].

3.2 Theorem 1

The optimal production rate vector

$$u^*(s) = [u_1^*(s), u_2^*(s), u_3^*(s), \ldots, u_n^*(s)], (t \le s \le t_f), \qquad (3.6)$$

belongs to the finite set of vectors.

$$\Omega^* = \{(0,0), (u^*,0), (d_1,0), (0,u_2), (0,d_2), (d_2,0), (0,u_3), (0,d_3), (d_3,0), \ldots, (0,u_n), (0,d_n) \qquad (3.7)$$

where

$u_i^*(s) = 0$

If the machine is not producing or undergoing a set of change to a product Type; the machine produces at the demand rate $(u_i^*(s) = di)$ only when $x_i(s) = 0 (i = 1,2,3,\ldots,n)$.

[Source: Modification of Theorem by Elhafsi and Bai (1996b)].

The proof of Theorem 1 is provided in Appendix A1.

3.3 Theorem 2

Without loss of generality, Let Product n, Product n-1, Product n-2, …, and Product 2 be such that $\gamma_n d_n \geq \gamma_{n-1} d_{n-1} \geq ... \gamma_2 d_2 \geq \gamma_1 d_1$. If the setup costs $k_1, k_2, k_3, ..., k_n$ are zero, then $Y_1^* = 0, Y_2^* = 0, Y_3^* = 0, ..., Y_n^* = 0$ if and only if $\ell + \ell_1 - 1 \geq 0$. If $\ell + \ell_1 - 1 < 0$, then $Y_1^* = 0, Y_2^* = 0, Y_3^* = 0, ..., Y_n^* = 0$ if and only if $\frac{\gamma_n d_n}{\gamma_{n-1} d_{n-1}} = \frac{\gamma_{n-1} d_{n-1}}{\gamma_{n-2} d_{n-2}} = ... = \frac{\gamma_2 d_2}{\gamma_1 d_1} \leq (1 - \ell - \ell_1)$.

[Source: Modification of Theorem by Elhafsi and Bai (1996b)]

See Appendix A2 for proof.

3.4 Corollary 1

Assuming that Product n, Product n-1, Product n-2,…, and Product 2 be the product that such that $\gamma_n d_n \geq \gamma_{n-1} d_{n-1} \geq ... \gamma_2 d_2 \geq \gamma_1 d_1$. If the setup costs $k_1, k_2, k_3, ..., k_n$ are non zero, then $Y_1^* = 0, Y_2^* = 0, Y_3^* = 0, ..., Y_n^* = 0$ if and only if

$$0 \leq k \leq \left(\frac{Y_0^2}{2} \right) \min\{(1 - \ell_n)(\gamma_n d_n - \gamma_{n-1} d_{n-1} - ... - \gamma_1 d_1) + \varsigma \gamma_{n-1} d_{n-1} - \varsigma \gamma_{n-2} d_{n-2} -, ..., \varsigma \gamma_1 d_1; \\ (\ell + \ell_1 + \ell_2 + \ell_3 + ... + \ell_{n-1} - 1)\gamma_n d_n + ... + ((\ell - \ell_2 - 1)\gamma_2 d_2 + (1 - \ell_1)\gamma_1 d_1)\}.$$

The following theorem shows that, when the setup costs are zero for all products, at most one Product Type will be produced at the demand rate, during the cyclic schedule.

[Source: Modification of Corollary by Elhafsi and Bai (1996b)]

3.5 Theorem 3

If the setup costs $k_1, k_2, k_3, ..., k_n$ are zero, then at least one of the $Y_i's$ will be equal to zero (this will be proved later in Appendix A3).

[Source: Modification of Theorem by Elhafsi and Bai (1996b)]

CHAPTER FOUR

SOLUTION OF STEADY STATE OF FORMULATED PROBLEM

4.0 Optimal Solution at Machine Steady State

In Elhafsi and Bai (1996a) and Onanaye, (2006), the authors studied a special version of the ELSP where they considered controllable production rates at the beginning as well as within a production run. They determined the optimal solution for the two-product and three-product cases respectively (steady state cases; less product Type, in this work) analytically.

The optimal solution consists of the time the machine spends producing at the demand rate and at the maximum rate as well as the maximum inventory and backlog levels.

Elhafsi and Bai (1996) showed that depending on the parameters, the cyclic schedule will have four different shapes in their case study of two products. Onanaye (2006) for three-product, shows that there would be six possible cyclic schedules; for multiple-product n, there will be $2n$ possible cyclic schedules. The following notations were introduced and defined as:

For product $i(I = 1,2,3,\ldots,n)$

t_i = time-spent producing at maximum rate within a cyclic schedule.

Υ_i = time-spent producing at demand rate within a cyclic schedule

S_i = maximum inventory level

s_i = maximum backlog level

$\gamma_i = c_i^+ c_i^- / (c_i^+ + c_i^-)$ cost factor

$\ell_i = d_i/U_i$ utilization factor of the machine by product *i*

$A_i = \gamma_i / 2d_i(1-\ell_i)$

$T = \sum_{i=1}^{i=n}(t_i + \Upsilon_i + \delta_i)$ length of the cyclic schedule

$\delta = \delta_1 + \delta_2 + \delta_3 + ... + \delta_n$ = total setup time during T

$k = k_1 + k_2 + k_3 + ... + k_n$ = total setup cost during T

$\ell = \ell_1 + \ell_2 + \ell_3 + ... + \ell_n$ = total utilization factor of the machine

$\alpha_i = (1 - \ell_i)/(1 - \ell)$.

It can be shown (see Elhafsi and Bai, 1996; Onanaye, 2006) that $S_1, S_2, S_3, ..., S_n, \Upsilon_1, \Upsilon_2, \Upsilon_3, ..., and\, \Upsilon_n$, are the solution of the following optimization problem:

Minimise $F(S, \Upsilon)$

$$= \sum_{i=1}^{i=n} (K_i + \tfrac{1}{2di(i-\ell_i)} (c_i^+ S_i^2 + c_i^- (S_i - Q_i^2)))/[\sum_{i=1}^{i=n} \tfrac{Q_i}{d_i} - \delta] \tag{4.10}$$

subject to

$$Q_i = q_i (1 + \sum_{i=1}^{i=n} (1 - \ell_j)\gamma_j / \delta - (1 - \ell)\gamma_i / \delta) \quad i = 1,2,3,...,n \tag{4.11}$$

$\delta \geq 0,\ \Upsilon_i \geq 0,\ Q_i \geq 0,\ i = 1, 2, 3, ..., n$

$Q_i = S_i - s_i$ and

$q_i = d_i \delta (1 - \ell_i)/(1 - \ell)$

The Optimal solution to this problem is given as follows:

by substituting S_i in $F(S, \Upsilon)$ we have

$$F(S, \Upsilon) = \sum_{i=1}^{i=n} (K_i + \tfrac{1}{2di(i-\ell_i)} (c_i^+ S_i^2 + c_i^- (S_i - Q_i)^2))/[\sum_{i=1}^{i=n} \tfrac{Q_i}{d_i} - \delta]$$

$$= \sum_{i=1}^{i=n} (K_i + \tfrac{1}{2di(i-\ell_i)} [c_i^+ (\frac{Q_i c_i^-}{2di(i - \ell_i)})^2 + c_i^- (\frac{Q_i c_i^-}{2di(i - \ell_i)} - Q_i)^2])/[\sum_{i=1}^{i=n} \tfrac{Q_i}{d_i} - \delta]$$

By further simplification we have

$$F(S, \Upsilon) = [k + A_1 Q_1^2 + A_2 Q_2^2 + A_3 Q_3^2 + ... + A_n Q_n^2]/[\tfrac{Q_1}{d_1} + \tfrac{Q_2}{d_2} + \tfrac{Q_3}{d_3} + ... + \tfrac{Q_n}{d_n} - \delta] \tag{4.12}$$

where

$k = k_1 + k_2 + k_3 + \ldots + k_n$ (total setup cost during T).

The quantities $Q_1, Q_2, Q_3, ..., Q_n, \Upsilon_1, \Upsilon_2, \Upsilon_3, ..., and\, \Upsilon_n$, are calculated under the following product-type as follows:

4.1 Two-Product Case

The optimal solution to this optimization problem is

$S_i = Q_i c_i^- /(c_i^+ + c_i^-),\ \ \mathrm{i} = 1,2$

$s_i = -Q_i c_i^- /(c_i^+ + c_i^-),\ \ \mathrm{i} = 1,2$

substituting S_i in $F(S, \Upsilon)$ gives

$$F(S, \Upsilon) = [k + A_1 Q_1^2 + A_2 Q_2^2] / [\tfrac{Q_1}{d_1} + \tfrac{Q_2}{d_2} - \delta] \tag{4.13}$$

$$Q_1^\mu = q_1 d_2 \gamma_2 (1 + \sqrt{1 + 2k(1-\ell)(\alpha_1 / \gamma_1 d_1 + \alpha_2 / \gamma_2 d_2) / \delta^2})) / (\alpha_2 / d_1 \gamma_1 + \alpha_1 / d_2 \gamma_2) \tag{4.14a}$$

$$Q_2^\mu = q_2 d_1 \gamma_1 (1 + \sqrt{1 + 2k(1-\ell)(\alpha_1 / d_1 \gamma_1 + \alpha_2 / d_2 \gamma_2) / \delta^2})) / (\alpha_2 / d_1 \gamma_1 + \alpha_1 / d_2 \gamma_2) \tag{4.14b}$$

$$\Upsilon_1^\mu = Q_2^\mu / d_2 - Q_1^\mu \rho_1 / (1 - \ell_1) d_1 - \delta\,; \tag{4.15a}$$

$$\Upsilon_2^\mu = Q_1^\mu / d_1 - Q_2^\mu \rho_2 / (1 - \ell_2) d_2 - \delta\,; \tag{4.15b}$$

If $\Upsilon_1^\mu \geq 0\ and\ \Upsilon_2^\mu \geq 0$ THEN

$\Upsilon_1^\mu = \Upsilon_1^*, \Upsilon_2^\mu = \Upsilon_2^*, Q_1^\mu = Q_1^*,\ and\ Q_2^\mu = Q_2^*$

STOP

ELSE

$$Q_1^\mu = (1 - \ell_1) d_1 \sqrt{k + A_2 (\delta d_2)^2) / [d_1^2 (1-\ell)^2 A_1 + d_2^2 \ell_1^2 A_2]} \tag{4.16a}$$

$$Q_2^\mu = d_2 \delta + \ell_1 d_2 \sqrt{k + A_2 (\delta d_2)^2) / [d_1^2 (1-\ell)^2 A_1 + d_2^2 \ell_1^2 A_2]} \tag{4.16b}$$

and compute $\Upsilon_1^\mu\ and\ \Upsilon_2^\mu\ u\sin g\ 4.15a\ and\ 4.15b$

If $\Upsilon_2^\mu > 0$ THEN

Calculate C2 = $F(\Upsilon_1^\mu, \Upsilon_2^\mu)$ using (4.13)

ELSE

C2 = ∞

END

Let

$$Q_1^{\mu} = d_1\delta + \ell_2 d_1\sqrt{k + A_1(\delta d_1)^2)/[d_2^2(1-\ell_2)^2 A_2 + d_1^2\ell_2^2 A_2]} \tag{4.17}$$

$$Q_2^{\mu} = (1-\ell_2)d_1\sqrt{k + A_1(\delta d_1)^2)/[d_2^2(1-\ell_2)^2 A_2 + d_1^2\ell_2^2 A_2]} \tag{4.18}$$

and compute Υ_1^{μ} *and* Υ_2^{μ} *using* 4.15*a and* 4.15*b*

If $\Upsilon_1^{\mu} > 0$ THEN

Calculate C1 = $F(\Upsilon_1^{\mu}, \Upsilon_2^{\mu})$ using (4.13)

ELSE

C1= ∞

END

LET

$Q_1^{\mu} = q_1, Q_2^{\mu} = q_2 \quad \rightarrow \Upsilon_1^{\mu} = 0,$ *and* $\Upsilon_2^{\mu} = 0$

Calculate $C0 = F(\Upsilon_1^{\mu}, \Upsilon_2^{\mu})$ (use 4.13)

$(\Upsilon_1^*, \Upsilon_2^*); (Q_1^*, Q_2^*) = \arg\min(\Upsilon_1^*, \Upsilon_2^*), (Q_1^*, Q_2^*)\{C0, C1, C2\}$

END IF

4.2 Three-Product Case

The optimal solution to this optimization problem is

$S_i = Q_i c_i^- /(c_i^+ + c_i^-),\ \ \mathrm{i} = 1,2,3$

$s_i = -Q_i c_i^- /(c_i^+ + c_i^-),\ \ \mathrm{i} = 1,2,3$

Substituting S_i in $F(S, \Upsilon)$ gives

$$F(S, \Upsilon) = [k + A_1 Q_1^2 + A_2 Q_2^2 + A_3 Q_3^2] / [\tfrac{Q_1}{d_1} + \tfrac{Q_2}{d_2} + \tfrac{Q_3}{d_3} - \delta] \tag{4.19}$$

$$Q_1^{\mu} = q_1 d_3 \gamma_3 (1 + \sqrt{1 + 2k(1-\ell)(\alpha_1/\gamma_1 d_1 + \alpha_3/\gamma_3 d_3)/\delta^2)}) / (\alpha_3/d_1\gamma_1 + \alpha_1/d_3\gamma_3) \tag{4.20a}$$

$$Q_2^{\mu} = q_2 d_1 \gamma_1 (1+\sqrt{1+2k(1-\ell)(\alpha_1/\gamma_1 d_1 + \alpha_2/\gamma_2 d_2)/\delta^2)})/(\alpha_2/d_1\gamma_1 + \alpha_1/d_2\gamma_2) \qquad (4.20b)$$

$$Q_3^{\mu} = q_3 d_2 \gamma_2 (1+\sqrt{1+2k(1-\ell)(\alpha_2/\gamma_2 d_2 + \alpha_3/\gamma_3 d_3)/\delta^2)})/(\alpha_3/\gamma_2 d_2 + \alpha_2/\gamma_3 d_3) \qquad (4.20c)$$

$$\Upsilon_1^{\mu} = Q_3^{\mu}/d_3 - Q_1^{\mu} p_1/(1-\ell_1)d_1 - \delta; \qquad (4.21a)$$

$$\Upsilon_2^{\mu} = Q_1^{\mu}/d_1 - Q_2^{\mu} p_2/(1-\ell_2)d_2 - \delta; \qquad (4.21b)$$

$$\Upsilon_3^{\mu} = Q_2^{\mu}/d_2 - Q_3^{\mu} p_2/(1-\ell_3)d_3 - \delta; \qquad (4.21c)$$

If $\Upsilon_1^{\mu} \geq 0,\ \Upsilon_2^{\mu} \geq 0,\ and\ \Upsilon_3^{\mu} \geq 0$ THEN

$\Upsilon_1^{*} = \Upsilon_1^{\mu}; \Upsilon_2^{*} = \Upsilon_2^{\mu}, and\ \Upsilon_3^{*} = \Upsilon_3^{\mu}, Q_1^{*} = Q_1^{\mu}, Q_2^{*} = Q_2^{\mu}, and\ Q_3^{*} = Q_3^{\mu},$

STOP

ELSE

$$Q_1^{\mu} = (1-\ell_1)d_1\sqrt{(k + A_3(\delta d_3)^2)/(d_1^2(1-\ell_1)^2 A_1 + d_2^2\ell_1^2 A_2 + d_3^2\ell_1^2 A_3)} \qquad (4.22a)$$

$$Q_2^{\mu} = d_2\delta + \ell_1 d_2\sqrt{(k + A_3(\delta d_3)^2)/(d_1^2(1-\ell_1)^2 A_1 + d_2^2\ell_1^2 A_2 + d_3^2\ell_1^2 A_3)} \qquad (4.22b)$$

$$Q_3^{\mu} = d_3\delta + \ell_1 d_3\sqrt{(k + A_3(\delta d_3)^2)/(d_1^2(1-\ell_1)^2 A_1 + d_2^2\ell_1^2 A_2 + d_3^2\ell_1^2 A_3)} \qquad (4.22c)$$

and compute $\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, and\ \Upsilon_3^{\mu}$ using (4.20a), (4.20b) and (4.20c)

If $\Upsilon_3^{\mu} > 0$ THEN

Calculate C3 = $F(\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, \Upsilon_3^{\mu})$ sing (4.19)

ELSE

C3 = ∞

END

Let

$$Q_1^{\mu} = d_1\delta + \ell_2 d_1\sqrt{(k + A_2(\delta d_2)^2)/(d_2^2(1-\ell_2)^2 A_2 + d_1^2\ell_2^2 A_1 + d_3^2\ell_2^2 A_3)} \qquad (4.23a)$$

$$Q_2^{\mu} = d_2\delta + \ell_2 d_1\sqrt{(k + A_2(\delta d_2)^2)/(d_2^2(1-\ell_2)^2 A_2 + d_1^2\ell_2^2 A_1 + d_3^2\ell_2^2 A_3)} \quad (4.23b)$$

$$Q_3^{\mu} = (1-\ell_1)d_3\sqrt{(k + A_2(\delta d_2)^2)/(d_2^2(1-\ell_2)^2 A_2 + d_1^2\ell_2^2 A_1 + d_3^2\ell_2^2 A_3)} \quad (4.23c)$$

and compute $\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, and\ \Upsilon_3^{\mu}$ using (4.20a), (4.20b) and (4.20c)

If $\Upsilon_2^{\mu} > 0$ THEN

Calculate C2 = $F(\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, \Upsilon_3^{\mu})$ using (4.19)

ELSE

C2 = ∞

END

Let

$$Q_1^{\mu} = d_1\delta + \ell_3 d_1\sqrt{(k + A_1(\delta d_1)^2)/(d_2^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_3^2\ell_3^2 A_3)} \quad (4.24a)$$

$$Q_2^{\mu} = (1-\ell_3)d_3\sqrt{(k + A_1(\delta d_1)^2)/(d_2^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_3^2\ell_3^2 A_3)} \quad (4.24b)$$

$$Q_3^{\mu} = d_1\delta + \ell_3 d_3\sqrt{(k + A_1(\delta d_1)^2)/(d_2^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_3^2\ell_3^2 A_3)} \quad (4.24c)$$

and compute $\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, and\ \Upsilon_3^{\mu}$ using (4.20a), (4.20b) and (4.20c)

If $\Upsilon_1^{\mu} > 0$ THEN

Calculate C1 = $F(\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, \Upsilon_3^{\mu})$ using (4.19)

ELSE

C1 = ∞

END

LET

$Q_1^{\mu} = q_1, Q_2^{\mu} = q_2, Q_3^{\mu} = q_3 \quad \rightarrow \Upsilon_1^{\mu} = 0, \Upsilon_2^{\mu} = 0\ and\ \Upsilon_3^{\mu} = 0$

Calculate $C0 = F(\Upsilon_1^{\mu}, \Upsilon_2^{\mu})$ (use 4.19)

$(Y_1^*, Y_2^*, Y_3^*); (Q_1^*, Q_2^*, Q_3^*) = \arg\min(\Upsilon_1^*, \Upsilon_2^*, \Upsilon_3^*), (Q_1^*, Q_2^*, Q_3^*)\{C0, C1, C2, C3\}$

END IF

4.3 Four-Product Case

The optimal solution to this optimization problem is

$S_i = Q_i c_i^- /(c_i^+ + c_i^-)$, i = 1,2,3,4

$s_i = -Q_i c_i^- /(c_i^+ + c_i^-)$, i = 1,2,3,4

Substituting S_i in $F(S,\Upsilon)$ gives

$$F(S,\Upsilon) = [k + A_1 Q_1^2 + A_2 Q_2^2 + A_3 Q_3^2 + A_4 Q_4^2]/[(Q_1/d_1) + (Q_2/d_2) + (Q_3/d_3) + (Q_4/d_4) - \delta] \quad (4.25)$$

where

$k = k_1 + k_2 + k_3 + k_4$ (total setup cost during time T)

The quantities $Q_1, Q_2, Q_3, Q_4,\ \Upsilon_1, \Upsilon_2, \Upsilon_3, and\, \Upsilon_4$ are calculated as

$$Q_1^u = q_1 d_4 \gamma_4 (1 + \sqrt{(1 + 2k(1-\ell)(\alpha_1/\gamma_1 d_1 + \alpha_4/\gamma_4 d_4)/\delta^2)})/(\alpha_4 d_1 \gamma_1 + \alpha_1 d_4 \gamma_4)$$

$$Q_2^u = q_2 d_1 \gamma_1 (1 + \sqrt{(1 + 2k(1-\ell)(\alpha_1/\gamma_1 d_1 + \alpha_2/\gamma_2 d_2)/\delta^2)})/(\alpha_2 d_1 \gamma_1 + \alpha_1 d_2 \gamma_3)$$

$$Q_3^u = q_3 d_2 \gamma_2 (1 + \sqrt{(1 + 2k(1-\ell)(\alpha_2/\gamma_2 d_2 + \alpha_3/\gamma_3 d_3)/\delta^2)})/(\alpha_3 d_2 \gamma_2 + \alpha_2 d_3 \gamma_3)$$

$$Q_4^u = q_4 d_3 \gamma_3 (1 + \sqrt{(1 + 2k(1-\ell)(\alpha_3/\gamma_3 d_3 + \alpha_4/\gamma_4 d_4)/\delta^2)})/(\alpha_3 d_4 \gamma_4 + \alpha_4 d_3 \gamma_3)$$

$$\Upsilon_1^\mu = Q_3^\mu / d_3 - Q_1^\mu \ell_1 / (1-\ell_1) d_1 - \delta \quad (4.26a)$$

$$\Upsilon_2^\mu = Q_1^\mu / d_1 - Q_2^\mu \ell_2 / (1-\ell_2) d_2 - \delta \quad (4.26b)$$

$$\Upsilon_3^\mu = Q_2^\mu / d_2 - Q_3^\mu \ell_3 / (1-\ell_3) d_3 - \delta \quad (4.26c)$$

$$\Upsilon_4^\mu = Q_3^\mu / d_3 - Q_4^\mu \ell_4 / (1-\ell_4) d_4 - \delta \quad (4.26d)$$

If $\Upsilon_1^\mu \geq 0,\ \Upsilon_2^\mu \geq 0, \Upsilon_3^\mu \geq 0$ and $\Upsilon_4^\mu \geq 0$ THEN

$\Upsilon_1^* = \Upsilon_1^\mu, \Upsilon_2^* = \Upsilon_2^\mu, \Upsilon_3^* = \Upsilon_3^\mu, \Upsilon_4^* = \Upsilon_4^\mu, Q_1^* = Q_1^\mu, Q_2^* = Q_2^\mu,\ Q_3^* = Q_3^\mu$, and $Q_4^* = Q_4^\mu$

STOP

ELSE

LET

$$Q_1^\mu = (1-\ell_1) d_1 \sqrt{((k + A_4(\delta d_4)^2)/(d_1^2(1-\ell_1)^2 A_1 + d_2^2 \ell_1^2 A_2 + d_3^2 \ell_1^2 A_3 + d_4^2 \ell_1^2 A_4))}$$

$$Q_2^\mu = d_2 \delta + \ell_1 d_2 \sqrt{((k + A_4(\delta d_4)^2)/(d_1^2(1-\ell_1)^2 A_1 + d_2^2 \ell_1^2 A_2 + d_3^2 \ell_1^2 A_3) + d_4^2 \ell_1^2 A_4\ )}$$

$$Q_3^\mu = d_3 \delta + \ell_1 d_3 \sqrt{((k + A_4(\delta d_4)^2)/(d_1^2(1-\ell_1)^2 A_1 + d_2^2 \ell_1^2 A_2 + d_3^2 \ell_1^2 A_3 + d_4^2 \ell_1^2 A_4))}$$

$$Q_4^\mu = d_4 \delta + \ell_1 d_4 \sqrt{((k + A_4(\delta d_4)^2)/(d_1^2(1-\ell_1)^2 A_1 + d_2^2 \ell_1^2 A_2 + d_3^2 \ell_1^2 A_3 + d_4^2 \ell_1^2 A_4))}$$

Compute $\Upsilon_1^\mu, \Upsilon_2^\mu, \Upsilon_3^\mu$ and Υ_4^μ (Use (4.26a), (4.26b), (4.26c) and (4.26d) respectively)

If $\Upsilon_4^\mu > 0$ THEN

Calculate $C4 = F(\Upsilon_1^\mu, \Upsilon_2^\mu, \Upsilon_3^\mu, \Upsilon_4^\mu)$ (use 4.25)

ELSE

C4 = ∞

END

LET

$$Q_1^\mu = d_1\delta + \ell_2 d_1\sqrt{\left((k + A_3(\delta d_3)^2)/(d_2^2(1-\ell_2)^2 A_2 + d_1^2\ell_2^2 A_1 + d_3^2\ell_2^2 A_3 + d_4^2\ell_2^2 A_4)\right)}$$
$$Q_2^\mu = (1-\ell_2)d_2\sqrt{\left((k + A_3(\delta d_3)^2)/(d_2^2(1-\ell_2)^2 A_2 + d_1^2\ell_2^2 A_1 + d_3^2\ell_2^2 A_3 + d_4^2\ell_2^2 A_4)\right)}$$
$$Q_3^\mu = d_3\delta + \ell_2 d_1\sqrt{\left((k + A_3(\delta d_3)^2)/(d_2^2(1-\ell_2)^2 A_2 + d_1^2\ell_2^2 A_1 + d_3^2\ell_2^2 A_3 + d_4^2\ell_2^2 A_4)\right)}$$
$$Q_4^\mu = d_4\delta + \ell_2 d_1\sqrt{\left((k + A_3(\delta d_3)^2)/(d_2^2(1-\ell_2)^2 A_2 + d_1^2\ell_2^2 A_1 + d_3^2\ell_2^2 A_3 + d_4^2\ell_2^2 A_4)\right)}$$

Recompute $\Upsilon_1^\mu, \Upsilon_2^\mu, \Upsilon_3^\mu$ and Υ_4^μ (Use (4.26a), (4.26b), (4.26c) and (4.26d) respectively)

If $\Upsilon_3^\mu > 0$ THEN

Calculate $C3 = F(\Upsilon_1^\mu, \Upsilon_2^\mu, \Upsilon_3^\mu, \Upsilon_4^\mu)$ (Use 4.25)

ELSE

C3 = ∞

END

LET

$$Q_1^\mu = d_1\delta + \ell_3 d_1\sqrt{\left((k + A_2(\delta d_2)^2)/(d_3^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_2^2\ell_3^2 A_3 + d_4^2\ell_3^2 A_4)\right)}$$
$$Q_2^\mu = d_3\delta + \ell_3 d_3\sqrt{\left((k + A_2(\delta d_2)^2)/(d_3^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_2^2\ell_3^2 A_3 + d_4^2\ell_3^2 A_4)\right)}$$
$$Q_3^\mu = (1-\ell_3)d_3\sqrt{\left((k + A_2(\delta d_2)^2)/(d_3^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_2^2\ell_3^2 A_3 + d_4^2\ell_3^2 A_4)\right)}$$
$$Q_4^\mu = d_4\delta + \ell_3 d_4\sqrt{\left((k + A_2(\delta d_2)^2)/(d_3^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_2^2\ell_3^2 A_3 + d_4^2\ell_3^2 A_4)\right)}$$

Recompute $\Upsilon_1^\mu, \Upsilon_2^\mu, \Upsilon_3^\mu$ and Υ_4^μ (Use (4.26a), (4.26b), (4.26c) and (4.26d) respectively)

If $\Upsilon_2^\mu > 0$ THEN

Calculate $C2 = F(\Upsilon_1^\mu, \Upsilon_2^\mu, \Upsilon_3^\mu, \Upsilon_4^\mu)$ (Use 4.25)

ELSE

C2 = ∞

END

LET

$$Q_1^{\mu} = d_1\delta + \ell_4 d_1\sqrt{\left((k + A_1(\delta d_1)^2)/(d_4^2(1-\ell_4)^2 A_3 + d_1^2\ell_4^2 A_1 + d_2^2\ell_4^2 A_3 + d_3^2\ell_4^2 A_4)\right)}$$
$$Q_2^{\mu} = d_2\delta + \ell_4 d_2\sqrt{\left((k + A_1(\delta d_1)^2)/(d_4^2(1-\ell_4)^2 A_3 + d_1^2\ell_4^2 A_1 + d_2^2\ell_4^2 A_3 + d_3^2\ell_4^2 A_4)\right)}$$
$$Q_3^{\mu} = d_3\delta + \ell_4 d_3\sqrt{\left((k + A_1(\delta d_1)^2)/(d_4^2(1-\ell_4)^2 A_3 + d_1^2\ell_4^2 A_1 + d_2^2\ell_4^2 A_3 + d_3^2\ell_4^2 A_4)\right)}$$
$$Q_4^{\mu} = (1-\ell_4) d_4\sqrt{\left((k + A_1(\delta d_1)^2)/(d_4^2(1-\ell_4)^2 A_3 + d_1^2\ell_4^2 A_1 + d_2^2\ell_4^2 A_3 + d_3^2\ell_4^2 A_4)\right)}$$

Recompute $\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, \Upsilon_3^{\mu}$ and Υ_4^{μ} (Use (4.26a), (4.26b), (4.26c) and (4.26d) respectively)

If $\Upsilon_1^{\mu} > 0$ THEN

Calculate $C1 = F(\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, \Upsilon_3^{\mu}, \Upsilon_4^{\mu})$ (Use (4.25))

ELSE

C1= ∞

END

LET

$Q_1^{\mu} = q_1, Q_2^{\mu} = q_2$,$Q_3^{\mu} = q_3$and $Q_4^{\mu} = q_4 \rightarrow \Upsilon_1^{\mu} = 0, \Upsilon_2^{\mu} = 0, \Upsilon_3^{\mu} = 0$ and $\Upsilon_4^{\mu} = 0$

Calculate $C0 = F(\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, \Upsilon_3^{\mu}, \Upsilon_4^{\mu})$ (use (4.25))

$$(\Upsilon_1^*, \Upsilon_2^*, \Upsilon_3^*, \Upsilon_4^*) : (Q_1^*, Q_2^*, Q_3^*, Q_4^*) = \arg\min(\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, \Upsilon_3^{\mu}, \Upsilon_4^{\mu}), (Q_1^{\mu}, Q_2^{\mu}, Q_3^{\mu}, Q_4^{\mu})\{C0, C1, C2, C3, C4\}$$

END IF

4.4 Five-Product Case

The optimal solution to this optimization problem is

$S_i = Q_i c_i^- /(c_i^+ + c_i^-)$, i = 1,2,3,4,5

$s_i = -Q_i c_i^- /(c_i^+ + c_i^-)$, i = 1,2,3,4,5

Substituting S_i in $F(S, \Upsilon)$ gives

$$F(S,\Upsilon) = \frac{[k + A_1 Q_1^2 + A_2 Q_2^2 + A_3 Q_3^2 + A_4 Q_4^2 + A_5 Q_5^2]}{[(Q_1/d_1) + (Q_2/d_2) + (Q_3/d_3) + (Q_4/d_4) + (Q_5/d_5) - \delta]} \quad (4.27)$$

where

$k = k_1 + k_2 + k_3 + k_4 + k_5$ (total setup cost during time T)

The quantities Q_1, Q_2, Q_3, Q_4, Q_5, $\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4, \Upsilon_5$ are calculated as

$$Q_1^{\mu}=q_1 d_5 \gamma_5 (1+\sqrt{(1+2k(1-\ell)(\alpha_1/\gamma_1 d_1+\alpha_5/\gamma_5 d_5)/\delta^2)})/(\alpha_5 d_1 \gamma_1+\alpha_1 d_5 \gamma_5)$$

$$Q_2^{\mu}=q_2 d_1 \gamma_1 (1+\sqrt{(1+2k(1-\ell)(\alpha_1/\gamma_1 d_1+\alpha_2/\gamma_2 d_2)/\delta^2)})/(\alpha_2 d_1 \gamma_1+\alpha_1 d_2 \gamma_3)$$

$$Q_3^{\mu}=q_3 d_2 \gamma_2 (1+\sqrt{(1+2k(1-\ell)(\alpha_2/\gamma_2 d_2+\alpha_3/\gamma_3 d_3)/\delta^2)})/(\alpha_3 d_2 \gamma_2+\alpha_2 d_3 \gamma_3)$$

$$Q_4^{\mu}=q_4 d_3 \gamma_3 (1+\sqrt{(1+2k(1-\ell)(\alpha_3/\gamma_3 d_3+\alpha_4/\gamma_4 d_4)/\delta^2)})/(\alpha_3 d_4 \gamma_4+\alpha_4 d_3 \gamma_3)$$

$$Q_5^{\mu}=q_5 d_4 \gamma_4 (1+\sqrt{(1+2k(1-\ell)(\alpha_4/\gamma_4 d_4+\alpha_5/\gamma_5 d_5)/\delta^2)})/(\alpha_4 d_5 \gamma_5+\alpha_5 d_4 \gamma_4)$$

$$\Upsilon_1^{\mu}=Q_3^{\mu}/d_3-Q_1^{\mu}\ell_1/(1-\ell_1)d_1-\delta \tag{4.28a}$$

$$\Upsilon_2^{\mu}=Q_1^{\mu}/d_1-Q_2^{\mu}\ell_2/(1-\ell_2)d_2-\delta \tag{4.28b}$$

$$\Upsilon_3^{\mu}=Q_2^{\mu}/d_2-Q_3^{\mu}\ell_3/(1-\ell_3)d_3-\delta \tag{4.28c}$$

$$\Upsilon_4^{\mu}=Q_3^{\mu}/d_3-Q_4^{\mu}\ell_4/(1-\ell_4)d_4-\delta \tag{4.28d}$$

$$\Upsilon_5^{\mu}=Q_4^{\mu}/d_4-Q_5^{\mu}\ell_5/(1-\ell_5)d_5-\delta \tag{4.28e}$$

If $\Upsilon_1^{\mu}\geq 0,\ \Upsilon_2^{\mu}\geq 0, \Upsilon_3^{\mu}\geq 0, \Upsilon_4^{\mu}\geq 0$ and $\Upsilon_5^{\mu}\geq 0$ THEN

$\Upsilon_1^{*}=\Upsilon_1^{\mu}, \Upsilon_2^{*}=\Upsilon_2^{\mu}, \Upsilon_3^{*}=\Upsilon_3^{\mu}, \Upsilon_4^{*}=\Upsilon_4^{\mu}, \Upsilon_5^{*}=\Upsilon_5^{\mu}, Q_1^{*}=Q_1^{\mu}, Q_2^{*}=Q_2^{\mu}, Q_3^{*}=Q_3^{\mu}, Q_4^{*}=Q_4^{\mu}, Q_5^{*}=Q_5^{\mu}$

STOP

ELSE

LET

$$Q_1^{\mu}=(1-\ell_1)d_1\sqrt{((k+A_5(\delta d_5)^2)/(d_1^2(1-\ell_1)^2A_1+d_2^2\ell_1^2A_2+d_3^2\ell_1^2A_3+d_4^2\ell_1^2A_4+d_5^2\ell_1^2A_5))}$$

$$Q_2^{\mu}=d_2\delta+\ell_1 d_2\sqrt{((k+A_5(\delta d_5)^2)/(d_1^2(1-\ell_1)^2A_1+d_2^2\ell_1^2A_2+d_3^2\ell_1^2A_3)+d_4^2\ell_1^2A_4+d_5^2\ell_1^2A_5\ )}$$

$$Q_3^{\mu}=d_3\delta+\ell_1 d_3\sqrt{((k+A_5(\delta d_5)^2)/(d_1^2(1-\ell_1)^2A_1+d_2^2\ell_1^2A_2+d_3^2\ell_1^2A_3+d_4^2\ell_1^2A_4+d_5^2\ell_1^2A_5))}$$

$$Q_4^{\mu}=d_4\delta+\ell_1 d_4\sqrt{((k+A_5(\delta d_5)^2)/(d_1^2(1-\ell_1)^2A_1+d_2^2\ell_1^2A_2+d_3^2\ell_1^2A_3+d_4^2\ell_1^2A_4+d_5^2\ell_1^2A_5))}$$

$$Q_5^{\mu}=d_5\delta+\ell_1 d_5\sqrt{((k+A_5(\delta d_5)^2)/(d_1^2(1-\ell_1)^2A_1+d_2^2\ell_1^2A_2+d_3^2\ell_1^2A_3+d_4^2\ell_1^2A_4+d_5^2\ell_1^2A_5))}$$

Compute $\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, \Upsilon_3^{\mu}, \Upsilon_4^{\mu}$ and Υ_5^{μ}

(Use (4.28a) , (4.28b) , (4.28c) , (4.28d) and (4.28e) respectively)

If $\Upsilon_5^{\mu}>0$ THEN

Calculate $C5=F(\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, \Upsilon_3^{\mu}, \Upsilon_4^{\mu}, \Upsilon_5^{\mu})$ (use 4.27)

ELSE

C5= ∞

END

LET

$$Q_1^\mu = d_1\delta + \ell_2 d_1\sqrt{\left((k + A_4(\delta d_4)^2)/(d_2^2(1-\ell_2)^2 A_2 + d_1^2\ell_2^2 A_1 + d_3^2\ell_2^2 A_3 + d_4^2\ell_2^2 A_4 + d_5^2\ell_2^2 A_5)\right)}$$
$$Q_2^\mu = (1-\ell_2)d_2\sqrt{\left((k + A_4(\delta d_4)^2)/(d_2^2(1-\ell_2)^2 A_2 + d_1^2\ell_2^2 A_1 + d_3^2\ell_2^2 A_3 + d_4^2\ell_2^2 A_4 + d_5^2\ell_2^2 A_5)\right)}$$
$$Q_3^\mu = d_3\delta + \ell_2 d_1\sqrt{\left((k + A_4(\delta d_4)^2)/(d_2^2(1-\ell_2)^2 A_2 + d_1^2\ell_2^2 A_1 + d_3^2\ell_2^2 A_3 + d_4^2\ell_2^2 A_4 + d_5^2\ell_2^2 A_5)\right)}$$
$$Q_4^\mu = d_4\delta + \ell_2 d_1\sqrt{\left((k + A_4(\delta d_4)^2)/(d_2^2(1-\ell_2)^2 A_2 + d_1^2\ell_2^2 A_1 + d_3^2\ell_2^2 A_3 + d_4^2\ell_2^2 A_4) + d_5^2\ell_2^2 A_5\right)}$$
$$Q_5^\mu = d_5\delta + \ell_2 d_1\sqrt{\left((k + A_4(\delta d_4)^2)/(d_2^2(1-\ell_2)^2 A_2 + d_1^2\ell_2^2 A_1 + d_3^2\ell_2^2 A_3 + d_4^2\ell_2^2 A_4 + d_5^2\ell_2^2 A_5)\right)}$$

Recompute $\Upsilon_1^\mu, \Upsilon_2^\mu, \Upsilon_3^\mu, \Upsilon_4^\mu$ and Υ_5^μ

(Use (4.28a) , (4.28b) , (4.28c) , (4.28d) and (4.28e) respectively)

If $\Upsilon_4^\mu > 0$ THEN

Calculate $C4 = F(\Upsilon_1^\mu, \Upsilon_2^\mu, \Upsilon_3^\mu, \Upsilon_4^\mu, \Upsilon_5^\mu)$ (Use 4.27)

ELSE

C4 = ∞

END

LET

$$Q_1^\mu = d_1\delta + \ell_3 d_1\sqrt{\left((k + A_3(\delta d_3)^2)/(d_3^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_2^2\ell_3^2 A_2 + d_4^2\ell_3^2 A_4) + d_5^2\ell_3^2 A_5)\right)}$$
$$Q_2^\mu = d_3\delta + \ell_3 d_3\sqrt{\left((k + A_3(\delta d_3)^2)/(d_3^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_2^2\ell_3^2 A_2 + d_4^2\ell_3^2 A_4 + d_5^2\ell_3^2 A_5)\right)}$$
$$Q_3^\mu = (1-\ell_3)d_3\sqrt{\left((k + A_3(\delta d_3)^2)/(d_3^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_2^2\ell_3^2 A_2 + d_4^2\ell_3^2 A_4 + d_5^2\ell_3^2 A_5)\right)}$$
$$Q_4^\mu = d_4\delta + \ell_3 d_4\sqrt{\left((k + A_3(\delta d_3)^2)/(d_3^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_2^2\ell_3^2 A_2 + d_4^2\ell_3^2 A_4 + d_5^2\ell_3^2 A_5)\right)}$$
$$Q_5^\mu = d_5\delta + \ell_4 d_5\sqrt{\left((k + A_3(\delta d_3)^2)/(d_3^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_2^2\ell_3^2 A_2 + d_4^2\ell_3^2 A_4 + d_5^2\ell_3^2 A_5)\right)}$$

Recompute $\Upsilon_1^\mu, \Upsilon_2^\mu, \Upsilon_3^\mu, \Upsilon_4^\mu$ and Υ_5^μ

(Use (4.28a), (4.28b), (4.28c), (4.28d) and (4.28e) respectively)

If $\Upsilon_3^\mu > 0$ THEN

Calculate $C3 = F(\Upsilon_1^\mu, \Upsilon_2^\mu, \Upsilon_3^\mu, \Upsilon_4^\mu, \Upsilon_5^\mu)$ (Use 4.27)

ELSE

C3 = ∞

END

LET

$$Q_1^\mu = d_1\delta + \ell_4 d_1\sqrt{\left((k + A_2(\delta d_2)^2)/(d_4^2(1-\ell_4)^2 A_4 + d_1^2\ell_4^2 A_1 + d_2^2\ell_4^2 A_2 + d_3^2\ell_4^2 A_3 + d_5^2\ell_4^2 A_5)\right)}$$
$$Q_2^\mu = d_2\delta + \ell_4 d_2\sqrt{\left((k + A_2(\delta d_2)^2)/(d_4^2(1-\ell_4)^2 A_4 + d_1^2\ell_4^2 A_1 + d_2^2\ell_4^2 A_2 + d_3^2\ell_4^2 A_3 + d_5^2\ell_4^2 A_5)\right)}$$
$$Q_3^\mu = d_3\delta + \ell_4 d_3\sqrt{\left((k + A_2(\delta d_2)^2)/(d_4^2(1-\ell_4)^2 A_4 + d_1^2\ell_4^2 A_1 + d_2^2\ell_4^2 A_2 + d_3^2\ell_4^2 A_3 + d_5^2\ell_4^2 A_5)\right)}$$
$$Q_4^\mu = (1-\ell_4)d_4\sqrt{\left((k + A_2(\delta d_2)^2)/(d_4^2(1-\ell_4)^2 A_4 + d_1^2\ell_4^2 A_1 + d_2^2\ell_4^2 A_2 + d_3^2\ell_4^2 A_3 + d_5^2\ell_4^2 A_5)\right)}$$
$$Q_5^\mu = d_5\delta + \ell_4 d_4\sqrt{\left((k + A_2(\delta d_2)^2)/(d_4^2(1-\ell_4)^2 A_4 + d_1^2\ell_4^2 A_1 + d_2^2\ell_4^2 A_2 + d_3^2\ell_4^2 A_3 + d_5^2\ell_4^2 A_5)\right)}$$

Recompute $\Upsilon_1^\mu, \Upsilon_2^\mu, \Upsilon_3^\mu, \Upsilon_4^\mu$ and Υ_5^μ

(Use (4.28a), (4.28b), (4.28c), (4.28d) and (4.28e) respectively)

If $\Upsilon_2^\mu > 0$ THEN

Calculate $C2 = F(\Upsilon_1^\mu, \Upsilon_2^\mu, \Upsilon_3^\mu, \Upsilon_4^\mu, \Upsilon_5^\mu)$ (Use 4.27)

ELSE

C2= ∞

END

LET

$$Q_1^\mu = d_1\delta + \ell_5 d_1\sqrt{\left((k + A_1(\delta d_1)^2)/(d_5^2(1-\ell_5)^2 A_5 + d_1^2\ell_5^2 A_1 + d_2^2\ell_5^2 A_3 + d_3^2\ell_5^2 A_3 + d_4^2\ell_5^2 A_4)\right)}$$
$$Q_2^\mu = d_2\delta + \ell_5 d_2\sqrt{\left((k + A_1(\delta d_1)^2)/(d_5^2(1-\ell_5)^2 A_5 + d_1^2\ell_5^2 A_1 + d_2^2\ell_5^2 A_3 + d_3^2\ell_5^2 A_3 + d_4^2\ell_5^2 A_4)\right)}$$
$$Q_3^\mu = d_3\delta + \ell_5 d_3\sqrt{\left((k + A_1(\delta d_1)^2)/(d_5^2(1-\ell_5)^2 A_5 + d_1^2\ell_5^2 A_1 + d_2^2\ell_5^2 A_3 + d_3^2\ell_5^2 A_3 + d_4^2\ell_5^2 A_4)\right)}$$
$$Q_4^\mu = d_4\delta + \ell_5 d_4\sqrt{\left((k + A_1(\delta d_1)^2)/(d_5^2(1-\ell_5)^2 A_5 + d_1^2\ell_{54}^2 A_1 + d_2^2\ell_5^2 A_3 + d_3^2\ell_5^2 A_3 + d_4^2\ell_5^2 A_4)\right)}$$
$$Q_5^\mu = (1-\ell_5)d_5\sqrt{\left((k + A_1(\delta d_1)^2)/(d_5^2(1-\ell_5)^2 A_5 + d_1^2\ell_5^2 A_1 + d_2^2\ell_5^2 A_3 + d_3^2\ell_5^2 A_3 + d_4^2\ell_5^2 A_4)\right)}$$

Recompute $\Upsilon_1^\mu, \Upsilon_2^\mu, \Upsilon_3^\mu, \Upsilon_4^\mu$ and Υ_5^μ

(Use (4.28a), (4.28b), (4.28c), (4.28d) and (4.28e) respectively)

If $\Upsilon_1^\mu > 0$ THEN

Calculate $C1 = F(\Upsilon_1^\mu, \Upsilon_2^\mu, \Upsilon_3^\mu, \Upsilon_4^\mu, \Upsilon_5^\mu)$ (Use 4.27)

LET

$Q_1^\mu = q_1, Q_2^\mu = q_2, Q_3^\mu = q_3, Q_4^\mu = q_4$ and $Q_5^\mu = q_5 \rightarrow \Upsilon_1^\mu = 0, \Upsilon_2^\mu = 0, \Upsilon_3^\mu = 0, \Upsilon_4^\mu = 0$ and $\Upsilon_5^\mu = 0$

Calculate $C0 = F(\Upsilon_1^\mu, \Upsilon_2^\mu, \Upsilon_3^\mu, \Upsilon_4^\mu, \Upsilon_5^\mu)$ (use 4.9)

$$(\Upsilon_1^*, \Upsilon_2^*, \Upsilon_3^*, \Upsilon_4^*, \Upsilon_5^*) : (Q_1^*, Q_2^*, Q_3^*, Q_4^*, Q_5^*) = \arg\min_{(Q_1^\mu, Q_2^\mu, Q_3^\mu, Q_4^\mu, Q_5^\mu)} (\Upsilon_1^\mu, \Upsilon_2^\mu, \Upsilon_3^\mu, \Upsilon_4^\mu, \Upsilon_5^\mu), \{C0, C1, C2, C3, C4, C5\}$$

END IF

4.5 n-Product Case

The optimal solution to the optimization problem is given as follows:

by substituting S_i in $F(S,\Upsilon)$ gives

$$F(S,\Upsilon) = \sum_{i=1}^{i=n}(K_i + \tfrac{1}{2di(i-\ell_i)}(c_i^+ S_i^2 + c_i^-(S_i - Q_i)^2))/[\sum_{i=1}^{i=n}\tfrac{Q_i}{d_i} - \delta]$$

$$= \sum_{i=1}^{i=n}(K_i + \tfrac{1}{2di(i-\ell_i)}[c_i^+(\frac{Q_i c_i^-}{2di(i-\ell_i)})^2 + c_i^-(\frac{Q_i c_i^-}{2di(i-\ell_i)} - Q_i)^2])/[\sum_{i=1}^{i=n}\tfrac{Q_i}{d_i} - \delta]$$

By further simplification we have

$$F(S,\Upsilon) = [k + A_1 Q_1^2 + A_2 Q_2^2 + A_3 Q_3^2 + \ldots + A_n Q_n^2]/[\tfrac{Q_1}{d_1} + \tfrac{Q_2}{d_2} + \tfrac{Q_3}{d_3} + \ldots + \tfrac{Q_n}{d_n} - \delta] \quad (4.29)$$

where

$k = k_1 + k_2 + k_3 + \ldots + k_n$ (total setup cost during T).

The quantities $Q_1, Q_2, Q_3, \ldots, Q_n, \Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n$ are calculated as follows:

$$Q_1^u = q_1 d_n \gamma_n (1 + \sqrt{(1 + 2k(1-\ell)(\alpha_1/\gamma_1 d_1 + \alpha_n/\gamma_n d_n)/\delta^2)})/(\alpha_n d_1 \gamma_1 + \alpha_1 d_n \gamma_n)$$
$$Q_2^u = q_2 d_1 \gamma_1 (1 + \sqrt{(1 + 2k(1-\ell)(\alpha_1/\gamma_1 d_1 + \alpha_2/\gamma_2 d_2)/\delta^2)})/(\alpha_2 d_1 \gamma_1 + \alpha_1 d_2 \gamma_3)$$
$$Q_3^u = q_3 d_2 \gamma_2 (1 + \sqrt{(1 + 2k(1-\ell)(\alpha_2/\gamma_2 d_2 + \alpha_3/\gamma_3 d_3)/\delta^2)})/(\alpha_3 d_2 \gamma_2 + \alpha_2 d_3 \gamma_3) \quad (4.30)$$
$$Q_4^u = q_4 d_3 \gamma_3 (1 + \sqrt{(1 + 2k(1-\ell)(\alpha_3/\gamma_3 d_3 + \alpha_4/\gamma_4 d_4)/\delta^2)})/(\alpha_3 d_4 \gamma_4 + \alpha_4 d_3 \gamma_3)$$

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$$Q_n^u = q_n d_{n-1} \gamma_{n-1} (1 + \sqrt{(1 + 2k(1-\ell)(\alpha_{n=1}/\gamma_{n-1} d_{n-1} + \alpha_n/\gamma_n d_n)/\delta^2)})/(\alpha_{n-1} d_n \gamma_n + \alpha_n d_{n-1} \gamma_{n-1})$$

$$\Upsilon_1^\mu = Q_n^\mu / d_n - Q_1^\mu \ell_1/(1-\ell_1)d_1 - \delta \quad (4.31a)$$
$$\Upsilon_2^\mu = Q_1^\mu / d_1 - Q_2^\mu \ell_2/(1-\ell_2)d_2 - \delta \quad (4.31b)$$
$$\Upsilon_3^\mu = Q_2^\mu / d_2 - Q_3^\mu \ell_3/(1-\ell_3)d_3 - \delta \quad (4.31c)$$

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$$\Upsilon_{n-1}^\mu = Q_{n-2}^\mu / d_{n-2} - Q_{n-1}^\mu \ell_{n-1}/(1-\ell_{n-1})d_{n-1} - \delta \quad (4.31y)$$
$$\Upsilon_n^\mu = Q_{n-1}^\mu / d_{n-1} - Q_n^\mu \ell_n/(1-\ell_n)d_n - \delta \quad (4.31z)$$

If $\Upsilon_1^\mu \ge 0,\ \Upsilon_2^\mu \ge 0, \Upsilon_3^\mu \ge 0, \ldots, \Upsilon_{n-1}^\mu \ge 0$ and $\Upsilon_n^\mu \ge 0$ THEN

$\Upsilon_1^* = \Upsilon_1^\mu, \Upsilon_2^* = \Upsilon_2^\mu, \Upsilon_3^* = \Upsilon_3^\mu, \ldots, \Upsilon_{n-1}^* = \Upsilon_{n-1}^\mu, \Upsilon_n^* = \Upsilon_n^\mu, Q_1^* = Q_1^\mu, Q_2^* = Q_2^\mu, Q_3^* = Q_3^\mu, \ldots, Q_{n-1}^* = Q_{n-1}^\mu, Q_n^* = Q_n^\mu$

STOP

ELSE

$$Q_1^{\mu} = (1-\ell_1)d_1\sqrt{\left((k + A_n(\delta d_n)^2)/(d_1^2(1-\ell_1)^2A_1 + d_2^2\ell_1^2A_2 + d_3^2\ell_1^2A_3 + \ldots + d_{n-1}^2\ell_1^2A_{n-1} + d_n^2\ell_1^2A_n)\right)}$$
$$Q_2^{\mu} = d_2\delta + \ell_1 d_2\sqrt{\left((k + A_n(\delta d_n)^2)/(d_1^2(1-\ell_1)^2A_1 + d_2^2\ell_1^2A_2 + d_3^2\ell_1^2A_3) + \ldots. + d_{n-1}^2\ell_1^2A_{n-1} + d_n^2\ell_1^2A_n\right)}$$
$$Q_3^{\mu} = d_3\delta + \ell_1 d_3\sqrt{\left((k + A_n(\delta d_n)^2)/(d_1^2(1-\ell_1)^2A_1 + d_2^2\ell_1^2A_2 + d_{n-2}^2\ell_1^2A_3 + \ldots + d_{n-1}^2\ell_1^2A_{n-1} + d_n^2\ell_1^2A_n)\right)}$$

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$$Q_{n-1}^{\mu} = d_{n-1}\delta + \ell_1 d_{n-1}\sqrt{\left((k + A_n(\delta d_n)^2)/(d_1^2(1-\ell_1)^2A_1 + d_2^2\ell_1^2A_2 + d_3^2\ell_1^2A_3 + \ldots + d_{n-1}^2\ell_1^2A_{n-1} + d_n^2\ell_1^2A_n)\right)}$$
$$Q_n^{\mu} = d_n\delta + \ell_1 d_n\sqrt{\left((k + A_n(\delta d_n)^2)/(d_1^2(1-\ell_1)^2A_1 + d_2^2\ell_1^2A_2 + d_3^2\ell_1^2A_3 + \ldots + d_{n-1}^2\ell_1^2A_{n-1} + d_n^2\ell_1^2A_n)\right)}$$

Compute $\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, \Upsilon_3^{\mu}, \ldots, \Upsilon_{n-1}^{\mu}$ and Υ_n^{μ}

(Use (4.31a), (4.31b), (4.31c),…, (4.31y) and (4.31z) respectively)

If $\Upsilon_n^{\mu} > 0$ THEN

Calculate $Cn = F(\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, \Upsilon_3^{\mu}, \ldots, \Upsilon_{n-1}^{\mu}, \Upsilon_n^{\mu})$ (use (4.29))

ELSE

Cn= ∞

END

LET

$$Q_1^{\mu} = d_1\delta + \ell_2 d_1\sqrt{\left((k + A_{n-1}(\delta d_{n-1})^2)/(d_2^2(1-\ell_2)^2A_2 + d_1^2\ell_2^2A_1 + d_3^2\ell_2^2A_3 + \ldots + d_{n-1}^2\ell_2^2A_{n-1} + d_n^2\ell_2^2A_n)\right)}$$
$$Q_2^{\mu} = (1-\ell_2)d_2\sqrt{\left((k + A_{n-1}(\delta d_{n-1})^2)/(d_2^2(1-\ell_2)^2A_2 + d_1^2\ell_2^2A_1 + d_3^2\ell_2^2A_3 + \ldots + d_{n-1}^2\ell_2^2A_{n-1} + d_n^2\ell_2^2A_n)\right)}$$
$$Q_3^{\mu} = d_3\delta + \ell_2 d_3\sqrt{\left((k + A_{n-1}(\delta d_{n-1})^2)/(d_2^2(1-\ell_2)^2A_2 + d_1^2\ell_2^2A_1 + d_3^2\ell_2^2A_3 + \ldots + d_{n-1}^2\ell_2^2A_{n-1} + d_n^2\ell_2^2A_n)\right)}$$

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$$Q_{n-1}^{\mu} = d_{n-1}\delta + \ell_2 d_{n-1}\sqrt{\left((k + A_{n-1}(\delta d_{n-1})^2)/(d_2^2(1-\ell_2)^2A_2 + d_1^2\ell_2^2A_1 + d_3^2\ell_2^2A_3 + \ldots + d_{n-1}^2\ell_2^2A_{n-1} + d_n^2\ell_2^2A_n)\right)}$$
$$Q_n^{\mu} = d_n\delta + \ell_2 d_n\sqrt{\left((k + A_{n-1}(\delta d_{n-1})^2)/(d_2^2(1-\ell_2)^2A_2 + d_1^2\ell_2^2A_1 + d_3^2\ell_2^2A_3 + \ldots + d_{n-1}^2\ell_2^2A_{n-1} + d_n^2\ell_2^2A_n)\right)}$$

Recompute $\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, \Upsilon_3^{\mu}, \ldots, \Upsilon_{n-1}^{\mu}$ and Υ_n^{μ}

(Use (4.31a), (4.31b), (4.31c),…, (4.31y) and (4.31z) respectively)

If $\Upsilon_{n-1}^{\mu} > 0$ THEN

Calculate $C(n-1) = F(\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, \Upsilon_3^{\mu}, \ldots, \Upsilon_{n-1}^{\mu}, \Upsilon_n^{\mu})$ (Use (4.29))

ELSE

Cn-1 = ∞

END

LET

$$Q_1^{\mu} = d_1\delta + \ell_3 d_1\sqrt{\left((k + A_{n-2}(\delta d_{n-2})^2)/(d_3^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_2^2\ell_3^2 A_2 + ... + d_{n-1}^2\ell_3^2 A_{n-1} + d_n^2\ell_3^2 A_n)\right)}$$
$$Q_2^{\mu} = d_3\delta + \ell_3 d_2\sqrt{\left((k + A_{n-2}(\delta d_{n-2})^2)/(d_3^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_2^2\ell_3^2 A_2 + ... + d_{n-1}^2\ell_3^2 A_{n-1} + d_n^2\ell_3^2 A_n)\right)}$$
$$Q_3^{\mu} = (1-\ell_3) d_3\sqrt{\left((k + A_{n-2}(\delta d_{n-2})^2)/(d_3^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_2^2\ell_3^2 A_2 + ... + d_{n-1}^2\ell_3^2 A_{n-1} + d_n^2\ell_3^2 A_n)\right)}$$
.
.
.
$$Q_{n-1}^{\mu} = d_{n-1}\delta + \ell_3 d_{n-1}\sqrt{\left((k + A_{n-2}(\delta d_{n-2})^2)/(d_3^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_2^2\ell_3^2 A_2 + ... + d_{n-1}^2\ell_3^2 A_{n-1} + d_n^2\ell_3^2 A_n)\right)}$$
$$Q_n^{\mu} = d_n\delta + \ell_3 d_n\sqrt{\left((k + A_{n-2}(\delta d_{n-2})^2)/(d_3^2(1-\ell_3)^2 A_3 + d_1^2\ell_3^2 A_1 + d_2^2\ell_3^2 A_2 + ... + d_{n-1}^2\ell_3^2 A_{n-1} + d_n^2\ell_3^2 A_n)\right)}$$

Recompute $\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, \Upsilon_3^{\mu}, ..., \Upsilon_{n-1}^{\mu}$ and Υ_n^{μ}

(Use (4.31a), (4.31b), (4.31c),…, (4.31y) and (4.31z) respectively)

If $\Upsilon_{n-2}^{\mu} > 0$ THEN

Calculate $C(n-2) = F(\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, \Upsilon_3^{\mu}, ..., \Upsilon_{n-1}^{\mu}, \Upsilon_n^{\mu})$ (Use (4.29))

ELSE

Cn-2 = ∞

END

We follow the same procedure for Cn-3, Cn-4, Cn-5,…, until we reach C2 and compute C2 as:

Let

$$Q_1^{\mu} = d_1\delta + \ell_{n-1} d_1\sqrt{\left((k + A_2(\delta d_2)^2)/(d_{n-1}^2(1-\ell_{n-1})^2 A_{n-1} + d_1^2\ell_{n-1}^2 A_1 + d_2^2\ell_{n-1}^2 A_2 + ... + d_{n-1}^2\ell_{n-1}^2 A_{n-1} + d_n^2\ell_{n-1}^2 A_n)\right)}$$
$$Q_2^{\mu} = d_2\delta + \ell_{n-1} d_2\sqrt{\left((k + A_2(\delta d_2)^2)/(d_{n-1}^2(1-\ell_{n-1})^2 A_{n-1} + d_1^2\ell_{n-1}^2 A_1 + d_2^2\ell_{n-1}^2 A_2 + ... + d_{n-1}^2\ell_{n-1}^2 A_{n-1} + d_n^2\ell_{n-1}^2 A_n)\right)}$$
$$Q_3^{\mu} = d_3\delta + \ell_{n-1} d_3\sqrt{\left((k + A_2(\delta d_2)^2)/(d_{n-1}^2(1-\ell_{n-1})^2 A_{n-1} + d_1^2\ell_{n-1}^2 A_1 + d_2^2\ell_{n-1}^2 A_2 + ... + d_{n-1}^2\ell_{n-1}^2 A_{n-1} + d_n^2\ell_{n-1}^2 A_n)\right)}$$
.
.
.
$$Q_{n-1}^{\mu} = (1-\ell_{n-1}) d_{n-1}\sqrt{\left((k + A_2(\delta d_2)^2)/(d_{n-1}^2(1-\ell_{n-1})^2 A_{n-1} + d_1^2\ell_{n-1}^2 A_1 + d_2^2\ell_{n-1}^2 A_2 + ... + d_{n-1}^2\ell_{n-1}^2 A_{n-1} + d_n^2\ell_{n-1}^2 A_n)\right)}$$
$$Q_n^{\mu} = d_n\delta + \ell_{n-1} d_n\sqrt{\left((k + A_2(\delta d_2)^2)/(d_{n-1}^2(1-\ell_{n-1})^2 A_{n-1} + d_1^2\ell_{n-1}^2 A_1 + d_2^2\ell_{n-1}^2 A_2 + ... + d_{n-1}^2\ell_{n-1}^2 A_{n-1} + d_n^2\ell_{n-1}^2 A_n)\right)}$$

Recompute $\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, \Upsilon_3^{\mu}, ..., \Upsilon_{n-1}^{\mu}$ and Υ_n^{μ}

(Use (4.31a), (4.31b), (4.31c),…, (4.31y) and (4.31z) respectively)

If $\Upsilon_2^{\mu} > 0$ THEN

Calculate $C2 = F(\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, \Upsilon_3^{\mu}, ..., \Upsilon_{n-1}^{\mu}, \Upsilon_n^{\mu})$ (Use 4.29)

ELSE

C2= ∞

END

LET

$$Q_1^{\mu} = d_1\delta + \ell_n d_1 \sqrt{\left((k + A_1(\delta d_1)^2)/(d_n^2(1-\ell_n)^2 A_n + d_1^2\ell_n^2 A_1 + d_2^2\ell_n^2 A_2 + \ldots + d_{n-1}^2\ell_n^2 A_{n-1} + d_n^2\ell_n^2 A_n)\right)}$$

$$Q_2^{\mu} = d_2\delta + \ell_n d_2 \sqrt{\left((k + A_1(\delta d_1)^2)/(d_n^2(1-\ell_n)^2 A_n + d_1^2\ell_n^2 A_1 + d_2^2\ell_n^2 A_2 + \ldots + d_{n-1}^2\ell_n^2 A_{n-1} + d_n^2\ell_n^2 A_n)\right)}$$

$$Q_3^{\mu} = d_3\delta + \ell_n d_3 \sqrt{\left((k + A_1(\delta d_1)^2)/(d_n^2(1-\ell_n)^2 A_n + d_1^2\ell_n^2 A_1 + d_2^2\ell_n^2 A_2 + \ldots + d_{n-1}^2\ell_n^2 A_{n-1} + d_n^2\ell_n^2 A_n)\right)}$$

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$$Q_{n-1}^{\mu} = d_{n-1}\delta + \ell_n d_{n-1} \sqrt{\left((k + A_1(\delta d_1)^2)/(d_n^2(1-\ell_n)^2 A_n + d_1^2\ell_n^2 A_1 + d_2^2\ell_n^2 A_2 + \ldots + d_{n-1}^2\ell_n^2 A_{n-1} + d_n^2\ell_n^2 A_n)\right)}$$

$$Q_n^{\mu} = (1-\ell_n) d_n \sqrt{\left((k + A_1(\delta d_1)^2)/(d_n^2(1-\ell_n)^2 A_n + d_1^2\ell_n^2 A_1 + d_2^2\ell_n^2 A_2 + \ldots + d_{n-1}^2\ell_n^2 A_{n-1} + d_n^2\ell_n^2 A_n)\right)}$$

Recompute $\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, \Upsilon_3^{\mu}, \ldots, \Upsilon_{n-1}^{\mu}$ and Υ_n^{μ}

(Use (4.31a), (4.31b), (4.31c),…, (4.31y) and (4.31z) respectively)

If $\Upsilon_1^{\mu} > 0$ THEN

Calculate $C1 = F(\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, \Upsilon_3^{\mu}, \ldots, \Upsilon_{n-1}^{\mu}, \Upsilon_n^{\mu})$ (Use (4.29))

LET

$Q_1^{\mu} = q_1, Q_2^{\mu} = q_2, Q_3^{\mu} = q_3, \ldots, Q_{n-1}^{\mu} = q_{n-1}$ and $Q_n^{\mu} = q_n \rightarrow \Upsilon_1^{\mu} = 0, \Upsilon_2^{\mu} = 0, \Upsilon_3^{\mu} = 0, \ldots, \Upsilon_{n-1}^{\mu} = 0$ and $\Upsilon_n^{\mu} = 0$

Calculate $C0 = F(\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, \Upsilon_3^{\mu}, \ldots, \Upsilon_{n-1}^{\mu}, \Upsilon_n^{\mu})$ (use (4.29))

$$(\Upsilon_1^*, \Upsilon_2^*, \Upsilon_3^*, \ldots, \Upsilon_{n-1}^*, \Upsilon_n^*) : (Q_1^*, Q_2^*, Q_3^*, \ldots, Q_{n-1}^*, Q_n^*) = \arg\min(\Upsilon_1^{\mu}, \Upsilon_2^{\mu}, \Upsilon_3^{\mu}, \ldots, \Upsilon_{n-1}^{\mu}, \Upsilon_n^{\mu}),$$
$$(Q_1^{\mu}, Q_2^{\mu}, Q_3^{\mu}, \ldots, Q_{n-1}^{\mu}, Q_n^{\mu})\{C0, C1, C2, C3, \ldots, Cn-1, Cn\}$$

END IF

CHAPTER FIVE

SOLUTION OF TRANSIENT STATE OF FORMULATED PROBLEM

5.0 Optimal Solution at Machine Transient State

As pointed out in Theorem 2, for a heavily loaded system (i.e. ℓ is close to one), the condition $\ell + \ell_1 + \ell_2 + \ell_3 + ... + \ell_{n-1} - 1) \geq 0$ will hold true. Hence $\Upsilon_1^*, \Upsilon_2^*, \Upsilon_3^*, ..., \Upsilon_n^*$ will be equal to zero.

5.1 Two-Product Case

In order to apply the result to other cases, we need to extract a schedule from the cyclic schedule, to obtain the optimal solution of the transient case. We divide the *x-space* into two mutually exclusive major regions $R^u \; and \; R^o$.

The Regions $R^u and \; R^o$ are defined as follows:

$$R^u = \begin{Bmatrix} (x_1, x_2) \mid d_2(1-\ell_2)(x_1 - A_1) + d_1\ell_2(x_2 - A_2) < 0; \\ d_2\ell_1(x_1 - C_1) + d_1(1-\ell_1)(x_2 - C_2) < 0 \end{Bmatrix}$$

$$R^o = \begin{Bmatrix} (x_1, x_2) \mid d_2(1-\ell_2)(x_1 - A_1) + d_1\ell_2(x_2 - A_2) \geq 0; \\ d_2\ell_1(x_1 - C_1) + d_1(1-\ell_1)(x_2 - C_2) \geq 0 \end{Bmatrix}$$

where A_1, A_2, C_1 and C_2 are defined as

$$\begin{aligned} A_1 &= S_1 + (d_1 / d_2)\ell_2\alpha_1 Q_2 - \alpha_1(1-\ell_2)(Q_1 - \delta d_1); \\ A_2 &= s_2 + \delta_2 d_2 + (d_2 / d_1)\ell_1\alpha_2(Q_1 - \delta d_1) - Q_2\ell_1\ell_2 / (1-\ell); \\ C_1 &= s_1 + \delta_1 d_1 + (d_1 / d_2)\ell_2\alpha_1(Q_2 - \delta d_2) - Q_1\ell_1\ell_2 / (1-\ell); \\ C_2 &= S_2 + (d_2 / d_1)\ell_1\alpha_2 Q_1 - \alpha_2(1-\ell_1)(Q_2 - \delta d_2) \end{aligned}$$

This is the case when there is production at the demand rate for both products (i.e. $\gamma_1 = 0 \, and \, \gamma_2 = 0, then \, Q_1 = q_1 \;\; and \, Q_2 = q_2$). However, if there is no production at the demand rate for both products, then A_1, A_2, C_1 and C_2 are defined as follows:

$$\begin{aligned} A_1 &= S_1; \\ A_2 &= s_2 + \delta_2 d_2; \\ C_1 &= s_1 + \delta_1 d_1; \\ C_2 &= S_2 \end{aligned}$$

We now carry out the transient analysis in two parts for each x – *space*.

In the first part, the optimal transient solution for all initial surplus levels in Region R^o was determined. In the second part, the optimal transient solution for initial surplus in Region R^u was derived. Without loss of generality, we shall index the product Types such that product Type 1 is the product Type with the largest setup time and setup cost, followed by product type 2 (i.e. $\delta_1 \geq \delta_2 \, and \, k_1 \geq k_2$).

5.1.2 Optimal Transient Solution in Region R^o

We applied the same techniques used in Bai and Elhafsi (1996) to determine the optimal solution for initial surplus levels in Region R^o. The region is partitioned into three mutually exclusive regions, T, T_1, T_2

Then regions T, T_1, T_2 are further partitioned into eight mutually exclusive sub-regions.

These sub-regions are $T_{11}, T_{12}, T_{21}, T_{22}, U_{11}, U_{12}, U_{21} \, and \, U_{22}$ for;

The regions are defined as follows:

5.1.3 Region R^o and its sub regions

$$T_{11} = \left\{ \begin{array}{l} (x_1, x_2) \mid x_1 - \delta_1 d_1 \geq 0; -d_2(x_1 - (s_1 + \delta_1 d_1)) + \\ d_2(x_2 - S_2) \geq 0 \end{array} \right\};$$

$$T_{12} = \left\{ \begin{array}{l} (x_1, x_2) \mid d_2(x_1 - (s_1 + \delta_1 d_1)) - d_1(x_2 - S_2) > 0; -d_2(x_1 - E_1) + \\ d_1(x_2 - E_2) \geq 0; d_2(1 - \ell_2)(x_1 - A_1) + d_1 \ell_2 (x_2 - A_2) \geq 0 \end{array} \right\};$$

$$T_{21} = \left\{ \begin{array}{l} (x_1, x_2) \mid -d_2(x_1 - S_1) + d_1(x_2 - (s_2 + \delta_2 d_2)) \geq 0; -d_2(x_1 - E_1) \\ -d_1(x_2 - E_2) > 0; d_2 \ell_1 (x_1 - C_1) + d_1(1 - \ell_1)(x_2 - C_2) \geq 0 \end{array} \right\};$$

$$T_{22} = \left\{ (x_1, x_2) \mid x_2 - \delta_2 d_2 \geq 0; d_2(x_1 - S_2) - d_1(x_2 - (s_2 + \delta_2 d_2) > 0 \right\};$$

$$U_{11} = \left\{ \begin{array}{l} (x_1, x_2) \mid x_1 - \delta_1 d_1 \geq 0; -d_2(x_1 - (s_1 + \delta_1 d_1)) + \\ d_2(x_2 - S_2) \geq 0; -d_2(x_1 - t_{11}) + d_1(x_2 - t_{12}) \geq 0; - \\ d_2(1 - \ell_2)(x_1 - A_1) - d_1 \ell_2 (x_2 - A_2) \geq 0 \end{array} \right\};$$

$$U_{12} = \left\{ \begin{array}{l} (x_1, x_2) \mid d_1\ell_1(x_1 - C_1); -d_2(1-\ell_1) + d_1(x_2 - C_2) > 0; - \\ d_2(x_1 - t_{11}) + d_1(x_2 - t_{12}) \geq 0; -d_2(1-\ell_2)(x_1 - A_1) - \\ d_1\ell_2(x_2 - A_2) \geq 0 \end{array} \right\};$$

$$U_{21} = \left\{ \begin{array}{l} (x_1, x_2) \mid d_2(x_1 - I_1) - d_1(x_2 - I_2) > 0; d_2(1-\ell_2)(x_1 - A_1) - d_1\ell_2(x_2 - A_2) \geq 0; - \\ d_2\ell_1(x_1 - C_1) - d_1(1-\ell_1)(x_2 - C_2) > 0; -d_2(x_1 - t_{21}) + d_1(x_2 - t_{22}) \geq 0 \end{array} \right\};$$

$$U_{22} = \left\{ \begin{array}{l} (x_1, x_2) \mid x_2 - \delta_2 d_2 \geq 0; -d_2(x_1 - S_1) - d_1(x_2 - (s_2 + \delta_2 d_2) > 0; -d_2\ell_1(x_1 - C_1) \\ -d_1(1-\ell_1)(x_2 - C_2) > 0; -d_2(x_1 - tt_{21}) + d_1(x_2 - t_{22}) > 0 \end{array} \right\};$$

$$T_1 = \left\{ \begin{array}{l} (x_1, x_2) \mid x_1 - \delta_1 d_1 > 0; d_2\ell_1(x_1 - (s_1 + \delta_1 d_1)) + \\ d_1(1-\ell_1)(x_2 - S_2) \geq 0 \end{array} \right\};$$

$$T_2 = \left\{ \begin{array}{l} (x_1, x_2) \mid -x_2 + \delta_2 d_2 > 0; d_2(1-\ell_2)(x_1 - S_1) + \\ d_1\ell_2(x_2 - (s_2 + \delta_2 d_2) \geq 0 \end{array} \right\};$$

where

$t_1 = (t_{11}, t_{12})^T$

is the point in *x-space* given by:

$t_{11} = \delta_1 d_1$

$t_{12} = C_2 + (C_1 - \delta_1 d_1) d_2 \ell_1 / d_1(1-\ell_1)$

$t_2 = (t_{21}, t_{22})^T$

is the point in *x-space* given by:

$t_{21} = A_1 + (A_2 - \delta_2 d_2) d_1 \ell_2 / d_2(1-\ell_2)$

$t_{22} = \delta_2 d_2$.

and

$E^{(1)} = (E_1^{(1)}, E_2^{(1)})^T$

is the point in *x-space* given by:

$E_1^{(1)} = (-\theta_2^{(1)} + \sqrt{\left((\theta_2^{(1)})^2 + \theta_1^{(1)}\theta_3^{(1)}\right)} / \theta_1^{(1)})$

and

$$E_2^{(1)} = \xi_1^{(1)}(-\theta_2^{(1)} + \sqrt{\left((\theta_2^{(1)})^2 + \theta_1^{(1)}\theta_3^{(1)}\right)} / \theta_1^{(1)}) + \xi_2^{(1)}$$

where

$$\theta_1^{(1)} = (c_1^+ / d_1)((\xi_1^{(1)})^2 - (\xi_3^{(1)})^2 + (c_2^+ / d_2)(1-(\xi_5^{(1)})^2);$$

$$\theta_2^{(1)} = (c_1^+ / d_1)((\xi_1^{(1)}\xi_1^{(1)} - \xi_3^{(1)}\xi_4^{(1)}(c_2^+ / d_2)\xi_5^{(1)}\xi_6^{(1)});$$

$$\theta_3^{(1)} = 2(k_1 - k_2) + (c_1^+ / d_1)((\xi_4^{(1)})^2 - (\xi_2^{(1)})^2 + (c_2^+ / d_2)((\xi_6^{(1)})^2);$$

$$\xi_1^{(1)} = -(d_1 / d_2)(1-\ell_1) / \ell_1;$$

$$\xi_2^{(1)} = A_1^{(1)} + A_2^{(1)}(d_1 / d_2)(1-\ell_1) / \ell_1;$$

$$\xi_3^{(1)} = -(d_1 / d_2)\ell_2 / \ell_1;$$

$$\xi_4^{(1)} = \ell_1 A_1^{(1)} - (\ell_2 / \ell_1)(1-\ell_1)A_2^{(1)} + (1-\ell_2)C_1^{(1)} + \ell_2(d_1 / d_2)C_2^{(1)};$$

$$\xi_1^{(1)} = (1-\ell_2) / \ell_1;$$

$$\xi_6^{(1)} = \ell_2 C_2^{(1)} - A_2^{(1)}(1-\ell_2)(1-\ell_1) / \ell_1 + (d_2 / d_1)(1-\ell_2)(C_1^{(1)} - A_1^{(1)});$$

5.1.4 Optimal Transient Solution in Region $\mathbf{R^u}$

The optimal solution for initial surplus levels in Region $\mathbf{R^u}$ is not obvious and is more involved mathematically. We established the following facts and definitions, based on the results established in Bai and Elhafsi (1996).

FACT 1: For initial surplus levels in Region R^u, the optimal way to move (advance) towards the cyclic schedule is by producing at maximum machine speed whenever it is possible.

That is, $U^* = (U_1, 0)\, or\, (0, U_2)$ if the machine is producing and $U^* = (0,0)$ if the machine is undergoing a setup change.

Therefore, given the current setup state, the direction of the surplus trajectory in R^u can be determined.

DEFINITION 1

A trajectory is said to be following Direction D, if it moves parallel to line L_i in the direction of increasing x_i. That is, the machine is producing product Type (i = 1,2) at maximum speed.

DEFINITION 2

Let a $D_i - n - step\, trajectory\,(i = 1,2; n > 1)$, a trajectory that performs alternately m_i setup change –production runs of product Type j and m_l setup change-production runs of product Type $l(j \neq l)$ with the initial setup change to product Type i and the case segment touching the cyclic schedule at any boundary say, B *or* D. If n is even then $m_i = m_j = m_l = n/2$.

If n is odd, then

$m_i =$ (n + 1)/2,

$m_j =$ (n - 1)/2.

FACT 2:

To reach the cyclic schedule in finite time, starting with initial surplus levels in Region R^u the trajectory leading to the cyclic schedule must touch the boundary $L_{j,i}(j,i = 1,2\, and\, i \neq j)$ of Region R^0, just before switching to product Type j and reaching the cyclic schedule.

In Elhafsi and Bai (1996b), it was established that the optimal switching policy is a special corridor policy with two windows. In the case where production at the demand rates is possible in the cyclic schedule, this result is still valid. Define by the pairs of points $(W_1^{(1)}, V_1^{(1)})$ and $(W_2^{(1)}, V_2^{(1)})$

using the technique developed in Elhafsi and Bai (1996b), the corridor boundaries and windows are determined as follows:

5.1.5 For Region R_1^u

Equations of boundaries $B_1^{(1)}\, and\, B_2^{(1)}$

$$B_1^{(1)} = M_1^{(1)} x_1 + M_2^{(1)} x_2 + M_3^{(1)} = 0; \qquad (5.30)$$

$$M_1^{(1)} = c_1^- / (1-\ell_1) - \ell_1\ell_2 c_1^+ / (1-\ell_1)(1-\ell) - (d_1/d_2)(\ell_1/\ell_2)\alpha_2 c_2^-; \tag{5.31a}$$

$$M_2^{(1)} = -(d_1/d_2)\ell_2 c_1^+ / (1-\ell) - c_1^+ / \ell_2 - \alpha_2(1-\ell_2)c_2^- / \ell_2; \tag{5.31b}$$

$$M_3^{(1)} = -(M_1^{(1)}C_1^{(1)} + M_2^{(1)}C_2^{(1)}). \tag{5.31c}$$

and

$$B_2^{(1)} = N_1^{(1)}x_1 + N_2^{(1)}x_2 + N_3^{(1)} = 0; \tag{5.32}$$

$$N_1^{(1)} = -(d_2/d_1)\ell_1 c_2^+ / (1-\ell) - c_1^+ / \ell_1 - \alpha_1(1-\ell_2)c_1^- / \ell_1; \tag{5.33a}$$

$$N_2^{(1)} = c_2^- / (1-\ell_2) - \ell_1\ell_2 c_2^+ / (1-\ell_2)(1-\ell) - (d_2/d_1)(\ell_2/\ell_1)\alpha_2 c_1^-; \tag{5.33b}$$

$$N_3^{(1)} = -(N_1^{(1)}A_1^{(1)} + N_2^{(1)}A_2^{(1)}). \tag{5.33c}$$

Corridor windows are:

Let

$$W_1^{(1)} = (w_{11}^{(1)}, w_{12}^{(1)}), V_1^{(1)} = (v_{11}^{(1)}, v_{12}^{(1)})$$

$$W_2^{(1)} = (w_{21}^{(1)}, w_{22}^{(1)}), V_2^{(1)} = (v_{21}^{(1)}, v_{22}^{(1)})$$

In *x-space*. $W_1^{(1)}$ satisfies the following system:

$$\begin{cases} a_1 w_{11}^2 + a_2 w_{11} + a_3 = 0 & ; \\ w_{12} = -(M_1/M_2)w_{11} - (M_3/M_2); \end{cases} \tag{5.34}$$

$$\begin{cases} v_{11} = w_{11} + \ell_1\ell_2(w_{11} - C_1)/(1-\ell) + (d_1/d_2)\ell_2(1-\ell_1) \\ (w_{12} - C_2)/(1-\ell); \\ v_{12} = C_2 + \ell_1\ell_2(C_2 - w_{12})/(1-\ell) + (d_2/d_1)\ell_1(1-\ell_2) \\ (C_1 - w_{11})/(1-\ell); \end{cases} \tag{5.35}$$

where

$$a_1^{(1)} = \frac{(\gamma_{11}^2 - 1)c_1^- - \eta_{11}^2 c_1^+}{2d_1(1-\ell_1)} + \frac{(\gamma_{12}^2 - (M_1/M_2)^2)c_2^+ - \eta_{12}^2 c_2^-}{2d_2(1-\ell_2)} \tag{5.36a}$$

$$a_2^{(1)} = \frac{(\gamma_{11}(\mu_{11} - \delta_1 d_1) + \delta_1 d_1)c_1^- - \eta_{11} v_{11} c_1^+}{d_1(1-\ell_1)} + \frac{(\gamma_{12}\mu_{12} - M_1 M_3 / M_2^2)c_2^+ - \eta_{12}(v_{12} - \delta_2 d_2)c_2^-}{d_2(1-\ell_2)} \quad (5.36b)$$

$$a_3^{(1)} = \frac{\mu_{11}(\mu_{11} - 2\delta_1 d_1)c_1^- - (v_{11}^2 - I_2^2)c_1^+}{2d_1(1-\ell_1)} + \frac{(\mu_{12}^2 - (M_3/M_2)^2 - I_2^2)c_2^+ - (v_{12} - \delta_2 d_2)^2 c_2^-}{2d_2(1-\ell_2)} - k_2 \quad (5.36c)$$

$$\eta_{11}^{(1)} = \ell_2 (d_1 / d_2)(M_1 / M_2)\alpha_1 - \ell_1 / (1-\ell) \quad (5.37a)$$

$$\eta_{12}^{(1)} = \alpha_2 \{(d_2 / d_1)\ell_1 - (M_1 / M_2)(1-\ell_1)\} \quad (5.37b)$$

$$v_{11}^{(1)} = \alpha_1 (1-\ell_2)A_1 + \alpha_1 \ell_2 (d_1 / d_2)(A_2 + (M_3 / M_2)) + \delta_1 d_1 \ell_2 / (1-\ell) \quad (5.38a)$$

$$v_{12}^{(1)} = -\ell_1 \alpha_2 (d_2 / d_1)A_1 - \ell_1 \ell_2 A_2 / (1-\ell) - (1-\ell_2)\alpha_1 (M_3 / M_2) - \alpha_2 (d_2 / d_1)\delta_1 d_1 \quad (5.38b)$$

$$\gamma_{11}^{(1)} = \alpha_1 \{(1-\ell_2) - \ell_2 (d_1 / d_2)(M_1 / M_2)\} \quad (5.39a)$$

$$\gamma_{12}^{(1)} = \ell_1 \ell_2 (M_1 / M_2) / (1-\ell) - \ell_1 \alpha_2 (d_2 / d_1) \quad (5.39b)$$

$$\mu_{11}^{(1)} = -\ell_1 \ell_2 C_2 / (1-\ell) - \ell_2 \alpha_1 (d_1 / d_2)C_2 + (M_3 / M_2)) \quad (5.40a)$$

$$\mu_{12}^{(1)} = \ell_1 \alpha_2 (d_2 / d_1)C_1 + (1-\ell_1)a_2 C_2 + \ell_1 \ell_2 (M_3 / M_2) / (1-\ell) \quad (5.40b)$$

and $W_2^{(1)}$ satisfies the following system:

$$\begin{cases} b_1 w_{22}^2 + b_2 w_{22} + b_3 = 0 \quad ; \\ w_{22} = -(N_2 / N_1) w_{22} - (N_3 / N_1); \end{cases} \quad (5.41)$$

$$\begin{cases} v_{21} = A_1 + \ell_1 \ell_2 (A_1 - w_{21}) / (1-\ell) + (d_1 / d_2)\ell_2 (1-\ell_1) \\ (A_2 - w_{22}) / (1-\ell); \\ v_{22} = w_{22} + \ell_1 \ell_2 (w_{22} - A_2) / (1-\ell) + (d_2 / d_1)\ell_1 (1-\ell_2) \\ (w_{21} - A_1) / (1-\ell); \end{cases} \quad (5.42)$$

where,

$$b_1^{(1)} = \frac{[((\gamma_{21}^{(1)})^2 - (N_2^{(1)} / N_1^{(1)})^2)c_1^+ - ((\eta_{21}^{(1)})^2 c_1^-)]}{2d_1(1-\ell_1)} + \frac{[((\gamma_{22}^{(1)})^2 c_2^- - (\eta_{22}^{(1)})^2 c_2^+)]}{2d_2(1-\ell_2)} \quad (5.43a)$$

$$b_2^{(1)} = \frac{(\gamma_{21}^{(1)}\mu_{21}^{(1)} - (N_2^{(1)} N_3^{(1)} / (N_1^{(1)})^2)c_1^+ - (\eta_{21}^{(1)})(v_{21} - \delta_1 d_1)c_1^-}{d_1(1-\ell_1)} + \frac{(\gamma_{22}^{(1)}(\mu_{22} - \delta_2 d_2) + \delta_2 d_2)c_2^- - \eta_{22} v_{22} c_1^-)]}{d_2(1-\ell_2)} \quad (5.43b)$$

$$b_3^{(1)} = \frac{[((\mu_{21}^{(1)})^2 - (N_3^{(1)}/N_1^{(1)})^2 - (I_1^{(1)})^2))c_1^+ - (v_{21}^{(1)} - \delta_1 d_1)^2 c_1^-]}{2d_1(1-\ell_1)} + \frac{\mu_{22}^{(1)}(\mu_{22}^{(1)} - 2\delta_2 d_2)^2 c_2^- - (v_{22}^{(1)})^2 - (I_2^{(1)})^2 c_2^+)]}{2d_2(1-\ell_2)} \quad (5.43c)$$

$$\eta_{21}^{(1)} = \alpha_1\{(d_1/d_2)\ell_2 - (N_2/N_1)(1-\ell_2)\}; \quad (5.44a)$$

$$\eta_{22}^{(1)} = \ell_1\alpha_2(d_2/d_1)(N_2/N_1) - \ell_1\ell_2/(1-\ell); \quad (5.44b)$$

$$v_{21}^{(1)} = -\ell_1\ell_2 C_1/(1-\ell) - \ell_2\alpha_1(d_1/d_2)C_2 - (1-\ell_2)\alpha_1(N_3/N_1) - \alpha_1(d_1/d_2)\delta_2 d_2; \quad (5.45a)$$

$$v_{22}^{(1)} = \ell_1\alpha_2(d_2/d_1)(C_1 + (N_3/N_1)) + (1-\ell_1)\alpha_2 C_2 + \delta_2 d_2 \ell_1/(1-\ell); \quad (5.45b)$$

$$\gamma_{21}^{(1)} = \ell_1\ell_2(N_2/N_1)/(1-\ell) - \ell_2\alpha_1(d_1/d_2); \quad (5.46a)$$

$$\gamma_{22}^{(1)} = \alpha_2\{(1-\ell) - \ell_1(d_2/d_1)(N_2/N_1)\}; \quad (5.46b)$$

$$\mu_{21}^{(1)} = (1-\ell_2)\alpha_1 A_1 + \ell_2\alpha_1(d_1/d_2)A_2 + \ell_2\ell_1(N_3/N_1)/(1-\ell); \quad (5.47a)$$

$$\mu_{22}^{(1)} = -\ell_1\alpha_2(d_1/d_2)(A_1 + (N_3/N_1)) - \ell_1\ell_2 A_2/(1-\ell). \quad (5.47b)$$

Notice that in (5.34), $w_{11}^{(1)}$ must be a real root of the quadratic such that $w_{11}^{(1)} \le C_1^{(1)}$.

Similarly, $w_{22}^{(1)}$ in (5.41) must be a real root such that $w_{22}^{(1)} \le A_2^{(1)}$. If either quadratic in (5.34) or (5.41) has no root, then the corridor window does not exist in this case meaning that it is always optimal to reach the extracted schedule form toe their side (i.e. the side with the corridor window).

5.1.6 Optimal Trajectory in Region R^u

Based on Facts 1 and 2, the Optimal Trajectory emanating from a point in Region R^u and leading to the extracted schedule in finite time can be obtained as follows: Given an initial surplus point in each of Region R^u ; the first setup is chosen and the cost of the trajectory leading to the extracted schedule with two setup changes only is calculated. At this point; in each case, there is a $D_i - 2 - step$ trajectory, where *i* is the initial setup for product Type *i*.

Next, attempt was made to lower the cost of the current trajectory by introducing a setup change to the other product Type so that the extracted schedule is reached at the opposite side. Therefore, if the cost can be reduced, then the extracted schedule is read at the opposite side. Therefore, if the cost can be reduced, that the obtained new trajectory is a $D_i - 3 - step$ trajectory. By keeping the trying to reduce the cost of the current trajectory by introducing each time, a setup change before the extracted schedule is reached until the cost cannot be lowered anymore. At this point, there is an optimal $D_i - n - step$ trajectory emanating in each Region R^u and reaching the extracted schedule in finite time. Similarly, the optimal $D_j - n - step$, where $(j \neq i)$, trajectories are obtained starting with a setup to product Type j first. Thus, the optimal trajectory would be the one with the lowest cost.

Bai and Elhafsi (1996) and Onanaye (2006), the numerical solutions of various examples, suggest that the above procedure be further simplified based on the following conjecture.

CONJECTURES

The initial setup of the optimal trajectory is given by $i \neq \arg\min_{i=1,2}\{C_i^2(x)\}$ where $C_i^2(x)$ is the cost of the $D_i - 2 - step$ trajectory staring with a setup change to product Type i and reaching the extracted schedule with two setup changes only. Based on the above Conjecture, the optimal trajectory emanating from $X = (a,b)$ in Region R^u is obtained using the following procedure.

PROCEDURE

STEP 0: $X_1(0) := a^{(1)}, X_2(0) := b^{(1)}; k := 1;$

$IF\ (C_1^{(1)})^2 X^{(1)}(0)) \geq (C_2^{(1)})^2 (X^{(1)}(0))$ THEN

$X_1(k) := X_1(0) - \delta_2 d_1;$

$X_2(k) := X_2(0) - \delta_2 d_2;$

GOTO STEP 1;

ELSE

$X_1(k) := X_1(0) - \delta_1 d_1;$

$X_2(k) := X_2(0) - \delta_1 d_2;$

ENDIF;

STEP 1: $k := k+1$

$X_1(k) := (d_1 \ell_2 M_3^{(1)} + X_2(k-1) M_2^{(1)} + d_2(1-\ell_2) X_1(k-1) M_2) / (d_2(1-\ell_2) M_2 - d_1 \ell_2 M_1);$

$X_2(k) := -(M_1^{(1)} / M_2^{(1)}) X_1(k) - (M_3^{(1)} / M_2^{(1)});$

IF $X_1(k) \geq w_{11}^{(1)}$ THEN

$X_1(k) := \gamma_{11}^{(1)} X_1(k-1) + \mu_{11}^{(1)}$;

$X_2(k) := \gamma_{12}^{(1)} X_2(k-1) + \mu_{12}^{(1)}$;

$k := k+1$

$X(k) := I^{(1)}$

GOTO STEP 3;

ELSE

GOTO STEP 2:

END IF;

STEP 2: $k := k+1$

$X_2(k) := (d_2\ell_1(N_3^{(1)} + X_1(k-1)N_1^{(1)} + d_1(1-\ell_1)X_1(k-1)N_1)/(d_1(1-\ell_1)N_1 - d_2\ell_1 N_2)$;

$X_1(k) := -(N_2^{(1)}/N_1^{(1)})X_2(k) - (N_3^{(1)}/N_1^{(1)})$;

IF $X_2(k) \geq w_{22}^{(1)}$ THEN

$X_1(k) := \gamma_{21}^{(1)} X_1(k-1) + \mu_{21}^{(1)}$;

$X_2(k) := \gamma_{22}^{(1)} X_2(k-1) + \mu_{22}^{(1)}$;

$k := k+1$

$X_1(k) := X_1(k-1) - \delta_2 d_1$;

$X_2(k) := X_2(k-1) - \delta_2 d_2$;

$k := k+1$

$X(k) := I^{(1)}$

GOTO STEP 3;

ELSE

GOTO STEP 1:

END IF;

STEP 3: OUTPUT: OPTIMAL TRAJECOTRY =

$\{X(0), X(1), \ldots, X(k-1), X(k)\}$;

5.2 Three-Product Case

In order to apply the result to other cases, we need to extract a schedule from the cyclic schedule, to obtain the optimal solution of the transient case. We divide the *x-space* into six mutually exclusive major regions (R_1^u, R_1^0 for (x_2, x_1)) pair, (R_2^u, R_2^0 for (x_3, x_2)) pair and (R_3^u, R_3^0 for (x_3, x_1)). We did this because, when production is switched from one product Type to another; say from product Type 1 to product Type 2, everything about production of product Type 3 is zero.

Therefore, it is more convenient to use (x_2, x_1), (x_3, x_2) and (x_3, x_1) x – spaces.

The Regions $R_1^u, R_1^0, R_2^u, R_2^0 \; and \; R_3^u, R_3^0$ are defined as follows:

$$R_1^u = \begin{Bmatrix} (x_1, x_2) \mid d_2(1-\ell_2)(x_1 - A_1^{(1)}) + d_1\ell_2(x_2 - A_2^{(1)}) < 0; \\ d_2\ell_1(x_1 - C_1^{(1)}) + d_1(1-\ell_1)(x_2 - C_2^{(1)}) < 0 \end{Bmatrix}$$

$$R_1^o = \begin{Bmatrix} (x_1, x_2) \mid d_2(1-\ell_2)(x_1 - A_1^{(1)}) + d_1\ell_2(x_2 - A_2^{(1)}) \geq 0; \\ d_2\ell_1(x_1 - C_1^{(1)}) + d_1(1-\ell_1)(x_2 - C_2^{(1)}) \geq 0 \end{Bmatrix}$$

where $A_1^{(1)}, A_2^{(1)}, C_1^{(1)} \; and \; C_2^{(1)}$ are defined as

$$\begin{aligned} A_1^{(1)} &= S_1 + (d_1 / d_2)\ell_2\alpha_1 Q_2 - \alpha_1(1-\ell_2)(Q_1 - \delta d_1); \\ A_2^{(1)} &= s_2 + \delta_2 d_2 + (d_2 / d_1)\ell_1\alpha_2(Q_1 - \delta d_1) - Q_2\ell_1\ell_2 / (1-\ell); \\ C_1^{(1)} &= s_1 + \delta_1 d_1 + (d_1 / d_2)\ell_2\alpha_1(Q_2 - \delta d_2) - Q_1\ell_1\ell_2 / (1-\ell); \\ C_2^{(1)} &= S_2 + (d_2 / d_1)\ell_1\alpha_2 Q_1 - \alpha_2(1-\ell_1)(Q_2 - \delta d_2) \end{aligned}$$

$$R_2^u = \begin{Bmatrix} (x_2, x_3) \mid d_3(1-\ell_3)(x_2 - A_1^{(2)}) + d_2\ell_3(x_3 - A_2^{(2)}) < 0; \\ d_3\ell_2(x_2 - C_1^{(2)}) + d_2(1-\ell_2)(x_3 - C_2^{(2)}) < 0 \end{Bmatrix}$$

$$R_2^o = \begin{Bmatrix} (x_2, x_3) \mid d_3(1-\ell_3)(x_2 - A_1^{(2)}) + d_2\ell_3(x_3 - A_2^{(2)}) \geq 0; \\ d_3\ell_2(x_2 - C_1^{(2)}) + d_2(1-\ell_2)(x_3 - C_2^{(2)}) \geq 0 \end{Bmatrix}$$

where $A_1^{(2)}, A_2^{(2)}, C_1^{(2)} \; and \; C_2^{(2)}$ are defined as

$$\begin{aligned} A_1^{(2)} &= S_2 + (d_2 / d_3)\ell_3\alpha_2 Q_3 - \alpha_2(1-\ell_3)(Q_2 - \delta d_2); \\ A_2^{(2)} &= s_3 + \delta_3 d_3 + (d_3 / d_2)\ell_2\alpha_3(Q_2 - \delta d_2) - Q_3\ell_2\ell_3 / (1-\ell); \\ C_1^{(2)} &= s_2 + \delta_2 d_2 + (d_2 / d_3)\ell_3\alpha_2(Q_3 - \delta d_2) - Q_2\ell_2\ell_3 / (1-\ell); \\ C_2^{(2)} &= S_3 + (d_3 / d_2)\ell_2\alpha_3 Q_2 - \alpha_3(1-\ell_2)(Q_3 - \delta d_3) \end{aligned}$$

$$R_3^u = \begin{Bmatrix} (x_1, x_3) \mid d_3(1-\ell_3)(x_1 - A_1^{(3)}) + d_1\ell_3(x_3 - A_2^{(3)}) < 0; \\ d_3\ell_1(x_1 - C_1^{(3)}) + d_1(1-\ell_1)(x_3 - C_2^{(3)}) < 0 \end{Bmatrix}$$

and

$$R_3^o = \left\{ \begin{matrix} (x_1, x_3) \mid d_3(1-\ell_3)(x_1 - A_1^{(3)}) + d_1\ell_3(x_3 - A_2^{(3)}) \geq 0; \\ d_3\ell_1(x_1 - C_1^{(3)}) + d_1(1-\ell_1)(x_3 - C_2^{(3)}) \geq 0 \end{matrix} \right\}$$

where $A_1^{(3)}, A_2^{(3)}, C_1^{(3)}$ *and* $C_2^{(3)}$ are defined as

$$\begin{aligned}
A_1^{(3)} &= S_1 + (d_1 / d_3)\ell_3\alpha_1 Q_3 - \alpha_1(1-\ell_3)(Q_1 - \delta d_1); \\
A_2^{(3)} &= s_3 + \delta_3 d_3 + (d_3 / d_1)\ell_1\alpha_3(Q_1 - \delta d_1) - Q_3\ell_1\ell_3 / (1-\ell); \\
C_1^{(3)} &= s_1 + \delta_1 d_1 + (d_1 / d_3)\ell_3\alpha_1(Q_3 - \delta d_1) - Q_1\ell_1\ell_3 / (1-\ell); \\
C_2^{(3)} &= S_3 + (d_3 / d_1)\ell_1\alpha_3 Q_1 - \alpha_3(1-\ell_1)(Q_3 - \delta d_3)
\end{aligned}$$

We now carry out the transient analysis in two parts for each $x - space$ pairs. In the first part, the optimal transient solution for all initial surplus levels in Regions $R^0(R_1^0, R_2^0, R_3^0)$ was determined. In the second part, the optimal transient solution for initial surplus in Regions $R^u(R_1^u, R_2^u, R_3^u)$ is derived. Without loss of generality, we shall index the product Types such that product Type 1 is the product Type with the largest setup time and setup cost, followed by product type 3 (i.e. $\delta_1 \geq \delta_2 \geq \delta_3$ and $k_1 \geq k_2 \geq k_3$)

5.2.1 Optimal Transient Solution in Region $R^0(R_1^0, R_2^0, R_3^0)$

We applied the same techniques used in Bai and Elhafsi (1996) and Onanaye (2006) to determine the optimal solution for initial surplus levels in Regions $R^0(R_1^0, R_2^0, R_3^0)$ Each of the regions is partitioned into three mutually exclusive regions, $T^{(1)}, T_1^{(1)}$ *and* $T_2^{(1)}$ *for* $R_1^0; T^{(2)}, T_1^{(2)}$ *and* $T_2^{(2)}$ *for* $R_2^0; T^{(3)}, T_1^{(3)}$ *and* $T_2^{(3)}$ *for* R_3^0.

Then regions $T^{(1)}, T^{(2)}$ *and* $T^{(3)}$ are further partitioned into eight mutually exclusive sub-regions.

These sub-regions are as follows:

$T_{11}^{(1)}, T_{12}^{(1)}, T_{21}^{(1)}, T_{22}^{(1)}, U_{11}^{(1)}, U_{12}^{(1)}, U_{21}^{(1)}$ *and* $U_{22}^{(1)})$ *for* $T^{(1)}$;

$T_{11}^{(2)}, T_{12}^{(2)}, T_{21}^{(2)}, T_{22}^{(2)}, U_{11}^{(2)}, U_{12}^{(2)}, U_{21}^{(2)}$ *and* $U_{22}^{(2)})$ *for* $T^{(2)}$;

$T_{11}^{(3)}, T_{12}^{(3)}, T_{21}^{(3)}, T_{22}^{(3)}, U_{11}^{(3)}, U_{12}^{(3)}, U_{21}^{(3)}$ *and* $U_{22}^{(3)})$ *for* $T^{(3)}$;

The regions are defined as follows:

5.2.2 Region R_1^0 and its sub regions

$$T_{11}^{(1)} = \left\{ \begin{matrix} (x_1, x_2) \mid x_1 - \delta_1 d_1 \geq 0; -d_2(x_1 - (s_1 + \delta_1 d_1)) + \\ d_2(x_2 - S_2) \geq 0 \end{matrix} \right\};$$

$$T_{12}^{(1)} = \left\{ \begin{matrix} (x_1, x_2) \mid d_2(x_1 - (s_1 + \delta_1 d_1)) - d_1(x_2 - S_2) > 0; -d_2(x_1 - E_1^{(1)}) + \\ d_1(x_2 - E_2^{(1)}) \geq 0; d_2(1 - \ell_2)(x_1 - A_1^{(1)}) + d_1 \ell_2 (x_2 - A_2^{(1)}) \geq 0 \end{matrix} \right\};$$

$$T_{21}^{(1)} = \left\{ \begin{matrix} (x_1, x_2) \mid -d_2(x_1 - S_1) + d_1(x_2 - (s_2 + \delta_2 d_2)) \geq 0; -d_2(x_1 - E_1^{(1)}) \\ -d_1(x_2 - E_2^{(1)}) > 0; d_2 \ell_1 (x_1 - C_1^{(1)}) + d_1(1 - \ell_1)(x_2 - C_2^{(1)}) \geq 0 \end{matrix} \right\};$$

$$T_{22}^{(1)} = \left\{ (x_1, x_2) \mid x_2 - \delta_2 d_2 \geq 0; d_2(x_1 - S_2) - d_1(x_2 - (s_2 + \delta_2 d_2) > 0 \right\};$$

$$U_{11}^{(1)} = \left\{ \begin{matrix} (x_1, x_2) \mid x_1 - \delta_1 d_1 \geq 0; -d_2(x_1 - (s_1 + \delta_1 d_1)) + \\ d_2(x_2 - S_2) \geq 0; -d_2(x_1 - t_{11}^{(1)}) + d_1(x_2 - t_{12}^{(1)}) \geq 0; - \\ d_2(1 - \ell_2)(x_1 - A_1^{(1)}) - d_1 \ell_2 (x_2 - A_2^{(1)}) \geq 0 \end{matrix} \right\};$$

$$U_{12}^{(1)} = \left\{ \begin{matrix} (x_1, x_2) \mid d_1 \ell_1 (x_1 - C_1^{(1)}); -d_2(1 - \ell_1) + d_1(x_2 - C_2^{(1)}) > 0; - \\ d_2(x_1 - t_{11}^{(1)}) + d_1(x_2 - t_{12}^{(1)}) \geq 0; -d_2(1 - \ell_2)(x_1 - A_1^{(1)}) - \\ d_1 \ell_2 (x_2 - A_2^{(1)}) \geq 0 \end{matrix} \right\};$$

$$U_{21}^{(1)} = \left\{ \begin{matrix} (x_1, x_2) \mid d_2(x_1 - I_1^{(1)}) - d_1(x_2 - I_2^{(1)}) > 0; d_2(1 - \ell_2)(x_1 - A_1^{(1)}) - d_1 \ell_2 (x_2 - A_2^{(1)}) \geq 0; - \\ d_2 \ell_1 (x_1 - C_1^{(1)}) - d_1(1 - \ell_1)(x_2 - C_2^{(1)}) > 0; -d_2(x_1 - t_{21}^{(1)}) + d_1(x_2 - t_{22}^{(1)}) \geq 0 \end{matrix} \right\};$$

$$U_{22}^{(1)} = \left\{ \begin{matrix} (x_1, x_2) \mid x_2 - \delta_2 d_2 \geq 0; -d_2(x_1 - S_1) - d_1(x_2 - (s_2 + \delta_2 d_2) > 0; -d_2 \ell_1 (x_1 - C_1^{(1)}) \\ -d_1(1 - \ell_1)(x_2 - C_2^{(1)}) > 0; -d_2(x_1 - t_{21}^{(1)}) + d_1(x_2 - t_{22}^{(1)}) > 0 \end{matrix} \right\};$$

$$T_1^{(1)} = \left\{ \begin{matrix} (x_1, x_2) \mid x_1 - \delta_1 d_1 > 0; d_2 \ell_1 (x_1 - (s_1 + \delta_1 d_1)) + \\ d_1(1 - \ell_1)(x_2 - S_2) \geq 0 \end{matrix} \right\};$$

$$T_2^{(1)} = \left\{ \begin{matrix} (x_1, x_2) \mid -x_2 + \delta_2 d_2 > 0; d_2(1 - \ell_2)(x_1 - S_1) + \\ d_1 \ell_2 (x_2 - (s_2 + \delta_2 d_2) \geq 0 \end{matrix} \right\};$$

where

$t_1^{(1)} = (t_{11}^{(1)}, t_{12}^{(1)})^T$

is the point in *x-space* given by:

$t_{11}^{(1)} = \delta_1 d_1$;

$t_{11}^{(1)} = C_2^{(1)} + (C_1^{(1)} - \delta_1 d_1)\delta_2 \ell_1 / d_1(1-\ell_1)$

$t_2^{(1)} = (t_{21}^{(1)}, t_{22}^{(1)})^T$;

is the point in *x-space* given by:

$t_{21}^{(1)} = A_1^{(1)} + (A_2^{(1)} - \delta_2 d_2) d_1 \ell_2 / d_2(1-\ell_2)$;

$t_{22}^{(1)} = \delta_2 d_2$;

and

$E^{(1)} = (E_1^{(1)}, E_2^{(1)})^T$

is the point in *x-space* given by:

$E_1^{(1)} = (-\theta_2^{(1)} + \sqrt{\left((\theta_2^{(1)})^2 + \theta_1^{(1)}\theta_3^{(1)}\right)} / \theta_1^{(1)})$

and

$E_2^{(1)} = \xi_1^{(1)}(-\theta_2^{(1)} + \sqrt{\left((\theta_2^{(1)})^2 + \theta_1^{(1)}\theta_3^{(1)}\right)} / \theta_1^{(1)}) + \xi_2^{(1)}$

where

$\theta_1^{(1)} = (c_1^+ / d_1)((\xi_1^{(1)})^2 - (\xi_3^{(1)})^2 + (c_2^+ / d_2)(1 - (\xi_5^{(1)})^2)$;

$\theta_2^{(1)} = (c_1^+ / d_1)((\xi_1^{(1)}\xi_1^{(1)} - \xi_3^{(1)}\xi_4^{(1)}(c_2^+ / d_2)\xi_5^{(1)}\xi_6^{(1)})$;

$\theta_3^{(1)} = 2(k_1 - k_2) + (c_1^+ / d_1)((\xi_4^{(1)})^2 - (\xi_2^{(1)})^2 + (c_2^+ / d_2)((\xi_6^{(1)})^2)$;

$\xi_1^{(1)} = -(d_1 / d_2)(1-\ell_1) / \ell_1$;

$\xi_2^{(1)} = A_1^{(1)} + A_2^{(1)}(d_1 / d_2)(1-\ell_1) / \ell_1$;

$\xi_3^{(1)} = -(d_1 / d_2)\ell_2 / \ell_1$;

$\xi_4^{(1)} = \ell_1 A_1^{(1)} - (\ell_2 / \ell_1)(1-\ell_1)A_2^{(1)} + (1-\ell_2)C_1^{(1)} + \ell_2(d_1 / d_2)C_2^{(1)}$;

$\xi_1^{(1)} = (1-\ell_2) / \ell_1$;

$$\xi_6^{(1)} = \ell_2 C_2^{(1)} - A_2^{(1)}(1-\ell_2)(1-\ell_1)/\ell_1 + (d_2/d_1)(1-\ell_2)(C_1^{(1)} - A_1^{(1)});$$

5.2.3 Region R_2^0 and its sub regions

$$T_{11}^{(2)} = \begin{Bmatrix} (x_2,x_3) \mid x_2 - \delta_2 d_2 \geq 0; -d_3(x_2 - (s_2 + \delta_2 d_2)) + \\ d_3(x_3 - S_3) \geq 0 \end{Bmatrix};$$

$$T_{12}^{(2)} = \begin{Bmatrix} (x_2,x_3) \mid d_3(x_2 - (s_2 + \delta_2 d_2)) - d_2(x_3 - S_3) > 0; -d_3(x_2 - E_1^{(2)}) + \\ d_3(x_3 - E_2^{(2)}) \geq 0; d_3(1-\ell_3)(x_2 - A_1^{(2)}) + d_2\ell_3(x_3 - A_2^{(2)}) \geq 0 \end{Bmatrix};$$

$$T_{21}^{(2)} = \begin{Bmatrix} (x_2,x_3) \mid -d_3(x_2 - S_2) + d_2(x_3 - (s_3 + \delta_3 d_3)) \geq 0; -d_3(x_2 - E_1^{(2)}) \\ -d_2(x_3 - E_2^{(2)}) > 0; d_3\ell_2(x_2 - C_1^{(2)}) + d_2(1-\ell_2)(x_3 - C_2^{(2)}) \geq 0 \end{Bmatrix};$$

$$T_{22}^{(2)} = \{(x_2,x_3) \mid x_3 - \delta_3 d_3 \geq 0; d_3(x_2 - S_3) - d_2(x_3 - (s_3 + \delta_3 d_3) > 0\};$$

$$U_{11}^{(2)} = \begin{Bmatrix} (x_2,x_3) \mid x_2 - \delta_2 d_2 \geq 0; -d_3(x_2 - (s_2 + \delta_2 d_2)) + \\ d_3(x_3 - S_3) \geq 0; -d_3(x_2 - t_{11}^{(2)}) + d_1(x_2 - t_{12}^{(2)}) \geq 0; - \\ d_3(1-\ell_3)(x_2 - A_1^{(2)}) - d_2\ell_3(x_3 - A_2^{(2)}) \geq 0 \end{Bmatrix};$$

$$U_{12}^{(2)} = \begin{Bmatrix} (x_2,x_3) \mid d_2\ell_2(x_2 - C_1^{(2)}); -d_3(1-\ell_2) + d_2(x_3 - C_2^{(2)}) > 0; - \\ d_3(x_2 - t_{11}^{(2)}) + d_2(x_3 - t_{12}^{(2)}) \geq 0; -d_3(1-\ell_3)(x_2 - A_1^{(2)}) - \\ d_2\ell_3(x_3 - A_2^{(2)}) \geq 0 \end{Bmatrix};$$

$$U_{21}^{(2)} = \begin{Bmatrix} (x_2,x_3) \mid d_3(x_2 - I_1^{(2)}) - d_2(x_3 - I_2^{(2)}) > 0; d_3(1-\ell_3)(x_2 - A_1^{(2)}) - d_1\ell_2(x_2 - A_2^{(2)}) \geq 0; - \\ d_3\ell_2(x_2 - C_1^{(2)}) - d_2(1-\ell_2)(x_3 - C_2^{(2)}) > 0; -d_3(x_2 - t_{21}^{(2)}) + d_2(x_3 - t_{22}^{(2)}) \geq 0 \end{Bmatrix};$$

$$U_{22}^{(2)} = \begin{Bmatrix} (x_2,x_3) \mid x_3 - \delta_3 d_3 \geq 0; -d_3(x_2 - S_2) - d_2(x_3 - (s_3 + \delta_3 d_3) > 0; -d_3\ell_2(x_2 - C_1^{(2)}) \\ -d_2(1-\ell_2)(x_3 - C_2^{(2)}) > 0; -d_3(x_2 - t_{21}^{(2)}) + d_2(x_3 - t_{22}^{(2)}) > 0 \end{Bmatrix};$$

$$T_1^{(2)} = \begin{Bmatrix} (x_2,x_3) \mid x_2 - \delta_2 d_2 > 0; d_3\ell_2(x_2 - (s_2 + \delta_2 d_2)) + \\ d_2(1-\ell_2)(x_3 - S_3) \geq 0 \end{Bmatrix};$$

$$T_2^{(2)} = \begin{Bmatrix} (x_2,x_3) \mid -x_3 + \delta_3 d_3 > 0; d_3(1-\ell_3)(x_2 - S_2) + \\ d_2\ell_3(x_3 - (s_3 + \delta_3 d_3) \geq 0 \end{Bmatrix};$$

where

$$t_1^{(2)} = (t_{11}^{(2)}, t_{12}^{(2)})^T$$

is the point in *x-space* given by:

$t_{11}^{(2)} = \delta_2 d_2$;

$t_{11}^{(2)} = C_2^{(2)} + (C_1^{(2)} - \delta_2 d_2)\delta_3 \ell_2 / d_2(1-\ell_2)$

$t_2^{(2)} = (t_{21}^{(2)}, t_{22}^{(2)})^T$;

is the point in *x-space* given by:

$t_{21}^{(2)} = A_1^{(2)} + (A_2^{(2)} - \delta_3 d_3) d_2 \ell_3 / d_3(1-\ell_3)$;

$t_{22}^{(2)} = \delta_3 d_3$;

and

$E^{(2)} = (E_1^{(2)}, E_2^{(2)})^T$

is the point in *x-space* given by:

$E_1^{(2)} = (-\theta_2^{(2)} + \sqrt{\left((\theta_2^{(2)})^2 + \theta_1^{(2)}\theta_3^{(2)}\right)} / \theta_1^{(2)})$

and

$E_2^{(2)} = \xi_1^{(2)}(-\theta_2^{(2)} + \sqrt{\left((\theta_2^{(2)})^2 + \theta_1^{(2)}\theta_3^{(2)}\right)} / \theta_1^{(2)}) + \xi_2^{(2)}$

where

$\theta_1^{(2)} = (c_2^+ / d_2)((\xi_1^{(2)})^2 - (\xi_3^{(2)})^2 + (c_3^+ / d_3)(1-(\xi_5^{(2)})^2)$;

$\theta_2^{(2)} = (c_2^+ / d_2)((\xi_1^{(2)}\xi_1^{(2)} - \xi_3^{(2)}\xi_4^{(2)}(c_3^+ / d_3)\xi_5^{(2)}\xi_6^{(2)})$;

$\theta_3^{(2)} = 2(k_2 - k_3) + (c_2^+ / d_2)((\xi_4^{(2)})^2 - (\xi_2^{(2)})^2 + (c_3^+ / d_3)((\xi_6^{(2)})^2)$;

$\xi_1^{(2)} = -(d_2 / d_3)(1-\ell_2) / \ell_2$;

$\xi_2^{(2)} = A_1^{(2)} + A_2^{(2)}(d_2 / d_3)(1-\ell_2) / \ell_2$;

$\xi_3^{(2)} = -(d_2 / d_3)\ell_3 / \ell_2$;

$\xi_4^{(2)} = \ell_2 A_1^{(2)} - (\ell_3 / \ell_2)(1-\ell_2) A_2^{(2)} + (1-\ell_3) C_1^{(2)} + \ell_3 (d_2 / d_3) C_2^{(2)}$;

$\xi_1^{(1)} = (1-\ell_2)/\ell_1;$

$\xi_6^{(1)} = \ell_2 C_2^{(1)} - A_2^{(1)}(1-\ell_2)(1-\ell_1)/\ell_1 + (d_2/d_1)(1-\ell_2)(C_1^{(1)} - A_1^{(1)});$

5.2.4 Region R_3^0 and its sub regions

$$T_{11}^{(3)} = \left\{ \begin{array}{l} (x_1, x_3) \mid x_1 - \delta_1 d_1 \geq 0; -d_3(x_1 - (s_1 + \delta_1 d_1)) + \\ d_3(x_3 - S_3) \geq 0 \end{array} \right\};$$

$$T_{12}^{(3)} = \left\{ \begin{array}{l} (x_1, x_3) \mid d_3(x_1 - (s_1 + \delta_1 d_1)) - d_1(x_3 - S_3) > 0; -d_3(x_1 - E_1^{(3)}) + \\ d_1(x_3 - E_2^{(3)}) \geq 0; d_3(1-\ell_3)(x_1 - A_1^{(3)}) + d_1 \ell_3 (x_3 - A_2^{(3)}) \geq 0 \end{array} \right\};$$

$$T_{21}^{(3)} = \left\{ \begin{array}{l} (x_1, x_3) \mid -d_3(x_1 - S_1) + d_1(x_3 - (s_3 + \delta_3 d_3)) \geq 0; -d_3(x_1 - E_1^{(3)}) \\ -d_1(x_3 - E_2^{(3)}) > 0; d_3 \ell_1 (x_1 - C_1^{(3)}) + d_1(1-\ell_1)(x_3 - C_2^{(3)}) \geq 0 \end{array} \right\};$$

$$T_{22}^{(3)} = \left\{ (x_1, x_3) \mid x_3 - \delta_3 d_3 \geq 0; d_3(x_1 - S_3) - d_1(x_3 - (s_3 + \delta_3 d_3) > 0 \right\};$$

$$U_{11}^{(3)} = \left\{ \begin{array}{l} (x_1, x_3) \mid x_1 - \delta_1 d_1 \geq 0; -d_3(x_1 - (s_1 + \delta_1 d_1)) + \\ d_3(x_3 - S_3) \geq 0; -d_3(x_1 - t_{11}^{(3)}) + d_1(x_3 - t_{12}^{(3)}) \geq 0; - \\ d_3(1-\ell_3)(x_1 - A_1^{(3)}) - d_1 \ell_3 (x_3 - A_2^{(3)}) \geq 0 \end{array} \right\};$$

$$U_{12}^{(3)} = \left\{ \begin{array}{l} (x_1, x_3) \mid d_1 \ell_1 (x_1 - C_1^{3)}); -d_3(1-\ell_1) + d_1(x_3 - C_2^{(3)}) > 0; - \\ d_3(x_1 - t_{11}^{(3)}) + d_1(x_3 - t_{12}^{(3)}) \geq 0; -d_3(1-\ell_3)(x_1 - A_1^{(3)}) - \\ d_1 \ell_3 (x_3 - A_2^{(3)}) \geq 0 \end{array} \right\};$$

$$U_{21}^{(3)} = \left\{ \begin{array}{l} (x_1, x_3) \mid d_3(x_1 - I_1^{(3)}) - d_1(x_3 - I_2^{(3)}) > 0; d_3(1-\ell_3)(x_1 - A_1^{(3)}) - d_1 \ell_3 (x_3 - A_2^{(3)}) \geq 0; - \\ d_3 \ell_1 (x_1 - C_1^{(3)}) - d_1(1-\ell_1)(x_3 - C_2^{(3)}) > 0; -d_3(x_1 - t_{21}^{(3)}) + d_1(x_3 - t_{22}^{(3)}) \geq 0 \end{array} \right\};$$

$$U_{22}^{(3)} = \left\{ \begin{array}{l} (x_1, x_3) \mid x_3 - \delta_3 d_3 \geq 0; -d_3(x_1 - S_1) - d_1(x_3 - (s_3 + \delta_3 d_3) > 0; -d_3 \ell_1 (x_1 - C_1^{(3)}) \\ -d_1(1-\ell_1)(x_3 - C_2^{(3)}) > 0; -d_3(x_1 - t_{21}^{(3)}) + d_1(x_3 - t_{22}^{(3)}) > 0 \end{array} \right\};$$

$$T_1^{(3)} = \left\{ \begin{array}{l} (x_1, x_3) \mid x_1 - \delta_1 d_1 > 0; d_3 \ell_1 (x_1 - (s_1 + \delta_1 d_1)) + \\ d_1(1-\ell_1)(x_3 - S_3) \geq 0 \end{array} \right\};$$

$$T_2^{(3)} = \left\{ \begin{array}{l} (x_1, x_3) \mid -x_3 + \delta_3 d_3 > 0; d_3(1-\ell_3)(x_1 - S_1) + \\ d_1 \ell_3 (x_3 - (s_3 + \delta_3 d_3) \geq 0 \end{array} \right\};$$

where

$t_1^{(3)} = (t_{11}^{(3)}, t_{12}^{(3)})^T$

is the point in *x-space* given by:

$t_{11}^{(3)} = \delta_1 d_1$;

$t_{11}^{(3)} = C_2^{(3)} + (C_1^{(3)} - \delta_1 d_1)\delta_3 \ell_1 / d_1(1-\ell_1)$

$t_2^{(3)} = (t_{21}^{(3)}, t_{22}^{(3)})^T$;

is the point in *x-space* given by:

$t_{21}^{(3)} = A_1^{(3)} + (A_2^{(3)} - \delta_3 d_3) d_1 \ell_3 / d_3(1-\ell_3)$;

$t_{22}^{(3)} = \delta_3 d_3$;

and

$E^{(3)} = (E_1^{(3)}, E_2^{(3)})^T$

is the point in *x-space* given by:

$E_1^{(3)} = (-\theta_2^{(3)} + \sqrt{\left((\theta_2^{(3)})^2 + \theta_1^{(3)}\theta_3^{(3)}\right)} / \theta_1^{(3)})$

and

$E_2^{(3)} = \xi_1^{(3)}(-\theta_2^{(3)} + \sqrt{\left((\theta_2^{(3)})^2 + \theta_1^{(3)}\theta_3^{(3)}\right)} / \theta_1^{(3)}) + \xi_2^{(3)}$

where

$\theta_1^{(3)} = (c_1^+ / d_1)((\xi_1^{(3)})^2 - (\xi_3^{(3)})^2 + (c_3^+ / d_3)(1-(\xi_5^{(3)})^2)$;

$\theta_2^{(3)} = (c_1^+ / d_1)((\xi_1^{(3)}\xi_1^{(3)} - \xi_3^{(3)}\xi_4^{(3)}(c_3^+ / d_3)\xi_5^{(3)}\xi_6^{(3)})$;

$\theta_3^{(3)} = 2(k_1 - k_3) + (c_1^+ / d_1)((\xi_4^{(3)})^2 - (\xi_2^{(3)})^2 + (c_3^+ / d_3)((\xi_6^{(3)})^2)$;

$\xi_1^{(3)} = -(d_1 / d_3)(1-\ell_1) / \ell_1$;

$\xi_2^{(3)} = A_1^{(3)} + A_2^{(3)}(d_1 / d_3)(1-\ell_1) / \ell_1$;

$\xi_3^{(3)} = -(d_1 / d_3)\ell_3 / \ell_1$;

$$\xi_4^{(3)} = \ell_1 A_1^{(3)} - (\ell_3 / \ell_1)(1-\ell_1)A_2^{(3)} + (1-\ell_3)C_1^{(3)} + \ell_3 (d_1 / d_3) C_2^{(3)};$$

$$\xi_1^{(3)} = (1-\ell_3)/\ell_1;$$

$$\xi_6^{(3)} = \ell_3 C_2^{(3)} - A_2^{(3)}(1-\ell_3)(1-\ell_1)/\ell_1 + (d_3/d_1)(1-\ell_3)(C_1^{(3)} - A_1^{(3)});$$

5.2.5 Optimal Transient Solution in Region $R^u(R_1^u, R_2^u, R_3^u)$

The optimal solution for initial surplus levels in Region $R^u(R_1^u, R_2^u, R_3^u)$ is not obvious and is more involved mathematically. We established the following facts and definitions, based on the results established in Bai and Elhafsi (1996) and Onanaye (2006).

FACT 1: For initial surplus levels in Region R^u the optimal way to move (advance) towards the cyclic schedule is by producing at maximum machine speed whenever it is possible.

That is, $U^* = (U_1, 0, 0)\, or\ (0, U_2, 0)\, or\ (0, 0, U_3)$ if the machine is producing and $U^* = (0,0,0)$ if the machine is undergoing a setup change.

Therefore, given the current setup state, the direction of the surplus trajectory in R^u can be known (or is known).

DEFINITION 1

A trajectory is said to be following Direction D_i, if it moves parallel to line L_i in the direction of increasing x_i. That is, the machine is producing product Type (i = 1,2,3) at maximum speed.

DEFINITION 2

Let a $D_i - n - step\, trajectory\,(1,2,3; n>1)$, a trajectory that performs alternately m_i setup change –production runs of product Type j and m_l setup change-production runs of product Type $l(j \neq l \neq i)$, with the initial setup change to product Type i and the case segment touching the cyclic schedule at any boundary. If n is even then $m_i = m_j = m_l = n/3$.

If n is odd, then

$m_i = (n + 1)/3,$

$m_j = (n - 1)/3,$

and

$m_l = (n - 1)/3.$

FACT 2:

To reach the cyclic schedule in finite time, starting with initial surplus levels in Region $R^u(R_1^u, R_2^u, R_3^u)$ the trajectory leading to the cyclic schedule must touch the boundary $L_{i,j,l}(i, j, l = 1,2,3 \textit{ and } i \neq j \neq l)$ of Region $R^u(R_1^u, R_2^u, R_3^u)$, just before switching to product Type *j* or to product Type *l* and reaching the cyclic schedule.

In Elhafsi and Bai (1996b) and Onanaye (2006), it has been established that the optimal switching policy is a special corridor policy with two windows. In the case where production at the demand rates is possible in the cyclic schedule, this result is still valid. Its corridor windows defined by the pairs of points $(W_1^{(1)}, V_1^{(1)})$ and $(W_2^{(1)}, V_2^{(1)})$; $(W_1^{(2)}, V_1^{(2)})$ and $(W_2^{(2)}, V_2^{(2)})$;$(W_1^{(3)}, V_1^{(3)})$ *and* $(W_2^{(3)}, V_2^{(3)})$ for each region.

Using the technique developed in Elhafsi and Bai (1996b) and Onanaye (2006), the corridor boundaries and windows are determined as follows:

5.2.6 For Region R_1^u

Equations of boundaries $B_1^{(1)} \textit{ and } B_2^{(1)}$:

$$B_1^{(1)} = M_1^{(1)}x_1 + M_2^{(1)}x_2 + M_3^{(1)} = 0; \tag{5.48}$$

$$M_1^{(1)} = c_1^- / (1-\ell_1) - \ell_1\ell_2 c_1^+ / (1-\ell_1)(1-\ell) - (d_1 / d_2)(\ell_1 / \ell_2)\alpha_2 c_2^-; \tag{5.49a}$$

$$M_2^{(1)} = -(d_1 / d_2)\ell_2 c_1^+ / (1-\ell) - c_1^+ / \ell_2 - \alpha_2(1-\ell_2)c_2^- / \ell_2; \tag{5.49b}$$

$$M_3^{(1)} = -(M_1^{(1)}C_1^{(1)} + M_2^{(1)}C_2^{(1)}). \tag{5.49c}$$

and

$$B_2^{(1)} = N_1^{(1)}x_1 + N_2^{(1)}x_2 + N_3^{(1)} = 0; \tag{5.50}$$

$$N_1^{(1)} = -(d_2/d_1)\ell_1 c_2^+/(1-\ell) - c_1^+/\ell_1 - \alpha_1(1-\ell_2)c_1^-/\ell_1; \tag{5.51a}$$

$$N_2^{(1)} = c_2^-/(1-\ell_2) - \ell_1\ell_2 c_2^+/(1-\ell_2)(1-\ell) - (d_2/d_1)(\ell_2/\ell_1)\alpha_2 c_1^-; \tag{5.51b}$$

$$N_3^{(1)} = -(N_1^{(1)}A_1^{(1)} + N_2^{(1)}A_2^{(1)}). \tag{5.51c}$$

Corridor windows are:

Let

$$W_1^{(1)} = (w_{11}^{(1)}, w_{12}^{(1)}), V_1^{(1)} = (v_{11}^{(1)}, v_{12}^{(1)})$$

$$W_2^{(1)} = (w_{21}^{(1)}, w_{22}^{(1)}), V_2^{(1)} = (v_{21}^{(1)}, v_{22}^{(1)})$$

in *x-space*. $W_1^{(1)}$ satisfies the following system:

$$\begin{cases} a_1^{(1)}(w_{11}^{(1)})^2 + a_2^{(1)}w_{11}^{(1)} + a_3^{(1)} = 0 & ; \\ w_{12}^{(1)} = -(M_1^{(1)}/M_2^{(1)})w_{11}^{(1)} - (M_3^{(1)}/M_2^{(1)}); \end{cases} \tag{5.52}$$

$$\begin{cases} v_{11}^{(1)} = w_{11}^{(1)} + \ell_1\ell_2(w_{11}^{(1)} - C_1^{(1)})/(1-\ell) + (d_1/d_2)\ell_2(1-\ell_1) \\ (w_{12}^{(1)} - C_2^{(1)})/(1-\ell); \\ v_{12}^{(1)} = C_2^{(1)} + \ell_1\ell_2(C_2^{(1)} - w_{12}^{(1)})/(1-\ell) + (d_2/d_1)\ell_1(1-\ell_2) \\ (C_1^{(1)} - w_{11}^{(1)})/(1-\ell); \end{cases} \tag{5.53}$$

where

$$a_1^{(1)} = \frac{((\gamma_{11}^{(1)})^2 - 1)c_1^- - (\eta_{11}^{(1)})^2 c_1^+}{2d_1(1-\ell_1)} + \frac{((\gamma_{12}^{(1)})^2 - (M_1^{(1)}/M_2^{(1)})^2)c_2^+ - (\eta_{12}^{(1)})^2 c_2^-}{2d_2(1-\ell_2)} \tag{5.54a}$$

$$a_2^{(1)} = \frac{(\gamma_{11}^{(1)}(\mu_{11}^{(1)} - \delta_1 d_1) + \delta_1 d_1)c_1^- - \eta_{11}^{(1)}v_{11}^{(1)}c_1^+}{d_1(1-\ell_1)} + \frac{(\gamma_{12}^{(1)}\mu_{12}^{(1)} - M_1^{(1)}M_3^{(1)}/(M_2^{(1)})^2)c_2^+ - \eta_{12}^{(1)}(v_{12}^{(1)} - \delta_2 d_2)c_2^-}{d_2(1-\ell_2)} \tag{5.54b}$$

$$a_3^{(1)} = \frac{\mu_{11}^{(1)}(\mu_{11}^{(1)} - 2\delta_1 d_1)c_1^- - ((v_{11}^{(1)})^2 - (I_2^{(1)})^2)c_1^+}{2d_1(1-\ell_1)} + \frac{((\mu_{12}^{(1)})^2 - (M_3^{(1)}/M_2^{(1)})^2 - (I_2^{(1)})^2)c_2^+ - (v_{12}^{(1)} - \delta_2 d_2)^2 c_2^-}{2d_2(1-\ell_2)} - k_2 \tag{5.54c}$$

$$\eta_{11}^{(1)} = \ell_2(d_1/d_2)(M_1^{(1)}/M_2^{(1)})\alpha_1 - \ell_1/(1-\ell) \tag{5.55a}$$

$$\eta_{12}^{(1)} = \alpha_2\{(d_2/d_1)\ell_1 - (M_1^{(1)}/M_2^{(1)})(1-\ell_1)\} \tag{5.55b}$$

$$v_{11}^{(1)} = \alpha_1(1-\ell_2)A_1^{(1)} + \alpha_1\ell_2(d_1/d_2)(A_2^{(1)} + (M_3^{(1)}/M_2^{(1)})) + \delta_1 d_1\ell_2/(1-\ell) \tag{5.56a}$$

$$v_{12}^{(1)} = -\ell_1\alpha_2(d_2/d_1)A_1^{(1)} - \ell_1\ell_2 A_2^{(1)}/(1-\ell) - (1-\ell_2)\alpha_1(M_3^{(1)}/M_2^{(1)}) - \alpha_2(d_2/d_1)\delta_1 d_1 \tag{5.56b}$$

$$\gamma_{11}^{(1)} = \alpha_1\{(1-\ell_2) - \ell_2(d_1/d_2)(M_1^{(1)}/M_2^{(1)})\} \quad (5.57a)$$

$$\gamma_{12}^{(1)} = \ell_1\ell_2(M_1^{(1)}/M_2^{(1)})/(1-\ell) - \ell_1\alpha_2(d_2/d_1) \quad (5.57b)$$

$$\mu_{11}^{(1)} = -\ell_1\ell_2 C_2/(1-\ell) - \ell_2\alpha_1(d_1/d_2)C_2^{(1)} + (M_3^{(1)}/M_2^{(1)})) \quad (5.58a)$$

$$\mu_{12}^{(1)} = \ell_1\alpha_2(d_2/d_1)C_1^{(1)} + (1-\ell_1)a_2C_2^{(1)} + \ell_1\ell_2(M_3^{(1)}/M_2^{(1)})/(1-\ell) \quad (5.58b)$$

and $W_2^{(1)}$ satisfies the following system:

$$\begin{cases} b_1^{(1)}(w_{22}^{(1)})^2 + b_2^{(1)}w_{22}^{(1)} + b_3^{(1)} = 0 \quad ; \\ w_{22}^{(1)} = -(N_2^{(1)}/N_1^{(1)})w_{22}^{(1)} - (N_3^{(1)}/N_1^{(1)}); \end{cases} \quad (5.59)$$

$$\begin{cases} v_{21}^{(1)} = A_1^{(1)} + \ell_1\ell_2(A_1^{(1)} - w_{21}^{(1)})/(1-\ell) + (d_1/d_2)\ell_2(1-\ell_1) \\ (A_2^{(1)} - w_{22}^{(1)})/(1-\ell); \\ v_{22}^{(1)} = w_{22}^{(1)} + \ell_1\ell_2(w_{22}^{(1)} - A_2^{(1)})/(1-\ell) + (d_2/d_1)\ell_1(1-\ell_2) \\ (w_{21}^{(1)} - A_1^{(1)})/(1-\ell); \end{cases} \quad (5.60)$$

where,

$$b_1^{(1)} = \frac{[((\gamma_{21}^{(1)})^2 - (N_2^{(1)}/N_1^{(1)})^2)c_1^+ - ((\eta_{21}^{(1)})^2 c_1^-)]}{2d_1(1-\ell_1)} + \frac{[((\gamma_{22}^{(1)})^2 c_2^- - (\eta_{22}^{(1)})^2 c_2^+)]}{2d_2(1-\ell_2)} \quad (5.61a)$$

$$b_2^{(1)} = \frac{(\gamma_{21}^{(1)}\mu_{21}^{(1)} - (N_2^{(1)}N_3^{(1)}/(N_1^{(1)})^2)c_1^+ - (\eta_{21}^{(1)})(v_{21}^{(1)} - \delta_1 d_1)c_1^-}{d_1(1-\ell_1)} + \frac{(\gamma_{22}^{(1)}(\mu_{22}^{(1)} - \delta_2 d_2) + \delta_2 d_2)c_2^- - \eta_{22}^{(1)}v_{22}^{(1)}c_1^-)]}{d_2(1-\ell_2)} \quad (5.61b)$$

$$b_3^{(1)} = \frac{[((\mu_{21}^{(1)})^2 - (N_3^{(1)}/N_1^{(1)})^2 - (I_1^{(1)})^2))c_1^+ - (v_{21}^{(1)} - \delta_1 d_1)^2 c_1^-]}{2d_1(1-\ell_1)} + \frac{\mu_{22}^{(1)}(\mu_{22}^{(1)} - 2\delta_2 d_2)^2 c_2^- - (v_{22}^{(1)})^2 - (I_2^{(1)})^2 c_2^+)]}{2d_2(1-\ell_2)} \quad (5.61c)$$

$$\eta_{21}^{(1)} = \alpha_1\left\{(d_1/d_2)\ell_2 - (N_2^{(1)}/N_1^{(1)})(1-\ell_2)\right\}; \quad (5.62a)$$

$$\eta_{22}^{(1)} = \ell_1\alpha_2(d_2/d_1)(N_2^{(1)}/N_1^{(1)}) - \ell_1\ell_2/(1-\ell); \quad (5.62b)$$

$$v_{21}^{(1)} = -\ell_1\ell_2 C_1^{(1)}/(1-\ell) - \ell_2\alpha_1(d_1/d_2)C_2^{(1)} - (1-\ell_2)\alpha_1(N_3^{(1)}/N_1^{(1)}) - \alpha_1(d_1/d_2)\delta_2 d_2; \quad (5.63a)$$

$$v_{22}^{(1)} = \ell_1\alpha_2(d_2/d_1)(C_1^{(1)} + (N_3^{(1)}/N_1^{(1)})) + (1-\ell_1)\alpha_2 C_2^{(1)} + \delta_2 d_2\ell_1/(1-\ell); \quad (5.63b)$$

$$\gamma_{21}^{(1)} = \ell_1\ell_2(N_2^{(1)}/N_1^{(1)})/(1-\ell) - \ell_2\alpha_1(d_1/d_2); \quad (5.64a)$$

$$\gamma_{22}^{(1)} = \alpha_2\{(1-\ell) - \ell_1(d_2/d_1)(N_2^{(1)}/N_1^{(1)})\}; \quad (5.64b)$$

$$\mu_{21}^{(1)} = (1-\ell_2)\alpha_1 A_1^{(1)} + \ell_2\alpha_1(d_1/d_2)A_2^{(1)} + \ell_2\ell_1(N_3^{(1)}/N_1^{(1)})/(1-\ell); \quad (5.65a)$$

$$\mu_{22}^{(1)} = -\ell_1\alpha_2(d_1/d_2)(A_1^{(1)} + (N_3^{(1)}/N_1^{(1)})) - \ell_1\ell_2 A_2^{(1)}/(1-\ell). \quad (5.65b)$$

Notice that in (5.52), $w_{11}^{(1)}$ must be a real root of the quadratic such that $w_{11}^{(1)} \le C_1^{(1)}$.

Similarly, $w_{22}^{(1)}$ in (5.59) must be a real root such that $w_{11}^{(1)} \le A_1^{(1)}$. If either quadratic in (5.52) or (5.59) has no root, then the corridor window does not exist in this case meaning that it is always optimal to reach the extracted schedule form toe their side (i.e. the side with the corridor window).

5.2.7 For Region R_2^u

Equations of boundaries $B_1^{(2)}$ *and* $B_2^{(2)}$:

$$B_1^{(2)} = M_1^{(2)}x_2 + M_2^{(2)}x_3 + M_3^{(2)} = 0; \quad (5.66)$$

$$M_1^{(2)} = c_2^-/(1-\ell_2) - \ell_2\ell_3 c_2^+/(1-\ell_2)(1-\ell) - (d_2/d_3)(\ell_2/\ell_3)\alpha_3 c_3^-; \quad (5.67a)$$

$$M_2^{(2)} = -(d_2/d_3)\ell_3 c_2^+/(1-\ell) - c_2^+/\ell_3 - \alpha_3(1-\ell_3)c_3^-/\ell_3; \quad (5.67b)$$

$$M_3^{(2)} = -(M_1^{(2)}C_1^{(2)} + M_2^{(2)}C_2^{(2)}). \quad (5.67c)$$

and

$$B_2^{(2)} = N_1^{(2)}x_2 + N_2^{(2)}x_3 + N_3^{(2)} = 0; \quad (5.68)$$

$$N_1^{(2)} = -(d_3/d_2)\ell_2 c_3^+/(1-\ell) - c_2^+/\ell_2 - \alpha_2(1-\ell_3)c_2^-/\ell_2; \quad (5.69a)$$

$$N_2^{(2)} = c_3^-/(1-\ell_3) - \ell_2\ell_3 c_3^+/(1-\ell_3)(1-\ell) - (d_3/d_2)(\ell_3/\ell_2)\alpha_3 c_2^-; \quad (5.69b)$$

$$N_3^{(2)} = -(N_1^{(2)}A_1^{(2)} + N_2^{(2)}A_2^{(2)}). \quad (5.69c)$$

Corridor windows are:

Let

$$W_1^{(2)} = (w_{11}^{(2)}, w_{12}^{(2)}), V_1^{(2)} = (v_{11}^{(2)}, v_{12}^{(2)})$$

$$W_2^{(2)} = (w_{21}^{(2)}, w_{22}^{(2)}), V_2^{(2)} = (v_{21}^{(2)}, v_{22}^{(2)})$$

in *x-space*. $W_1^{(2)}$ satisfies the following system:

$$\begin{cases} a_1^{(2)}(w_{11}^{(2)})^2 + a_2^{(2)} w_{11}^{(2)} + a_3^{(2)} = 0 \quad ; \\ w_{12}^{(2)} = -(M_1^{(2)} / M_2^{(2)}) w_{11}^{(2)} - (M_3^{(2)} / M_2^{(2)}); \end{cases} \tag{5.70}$$

$$\begin{cases} v_{11}^{(2)} = w_{11}^{(2)} + \ell_2 \ell_3 (w_{11}^{(2)} - C_1^{(2)}) / (1-\ell) + (d_2 / d_3) \ell_3 (1-\ell_2) \\ (w_{12}^{(2)} - C_2^{(2)}) / (1-\ell); \\ v_{12}^{(2)} = C_2^{(2)} + \ell_2 \ell_3 (C_2^{(2)} - w_{12}^{(2)}) / (1-\ell) + (d_3 / d_2) \ell_2 (1-\ell_3) \\ (C_1^{(2)} - w_{11}^{(2)}) / (1-\ell); \end{cases} \tag{5.71}$$

where

$$a_1^{(2)} = \frac{((\gamma_{11}^{(2)})^2 - 1) c_2^- - (\eta_{11}^{(2)})^2 c_2^+}{2 d_2 (1-\ell_2)} + \frac{((\gamma_{12}^{(2)})^2 - (M_1^{(1)} / M_2^{(1)})^2) c_3^+ - (\eta_{12}^{(1)})^2 c_3^-}{2 d_3 (1-\ell_3)} \tag{5.72a}$$

$$a_2^{(2)} = \frac{(\gamma_{11}^{(2)} (\mu_{11}^{(2)} - \delta_2 d_2) + \delta_2 d_2) c_2^- - \eta_{11}^{(2)} v_{11}^{(2)} c_2^+}{d_2 (1-\ell_2)} + \frac{(\gamma_{12}^{(2)} \mu_{12}^{(2)} - M_1^{(2)} M_3^{(2)} / (M_2^{(2)})^2) c_3^+ - \eta_{12}^{(2)} (v_{12}^{(2)} - \delta_3 d_3) c_3^-}{d_3 (1-\ell_3)} \tag{5.72b}$$

$$a_3^{(2)} = \frac{\mu_{11}^{(2)} (\mu_{11}^{(2)} - 2\delta_2 d_2) c_2^- - ((v_{11}^{(2)})^2 - (I_2^{(2)})^2) c_2^+}{2 d_2 (1-\ell_2)} + \frac{((\mu_{12}^{(2)})^2 - (M_3^{(2)} / M_2^{(2)})^2 - (I_2^{(2)})^2) c_3^+ - (v_{12}^{(2)} - \delta_3 d_3)^2 c_3^-}{2 d_3 (1-\ell_3)} - k_2 \tag{5.72c}$$

$$\eta_{11}^{(2)} = \ell_3 (d_2 / d_3)(M_1^{(2)} / M_2^{(2)}) \alpha_2 - \ell_2 / (1-\ell) \tag{5.73a}$$

$$\eta_{12}^{(2)} = \alpha_3 \{ (d_3 / d_2) \ell_2 - (M_1^{(2)} / M_2^{(2)})(1-\ell_2) \} \tag{5.73b}$$

$$v_{11}^{(2)} = \alpha_2 (1-\ell_3) A_1^{(2)} + \alpha_2 \ell_3 (d_2 / d_3)(A_2^{(2)} + (M_3^{(2)} / M_2^{(2)})) + \delta_2 d_2 \ell_3 / (1-\ell) \tag{5.74a}$$

$$v_{12}^{(2)} = -\ell_2 \alpha_3 (d_3 / d_2) A_1^{(2)} - \ell_2 \ell_3 A_2^{(2)} / (1-\ell) - (1-\ell_3) \alpha_2 (M_3^{(2)} / M_2^{(2)}) - \alpha_3 (d_3 / d_2) \delta_2 d_2 \tag{5.74b}$$

$$\gamma_{11}^{(2)} = \alpha_2 \{ (1-\ell_3) - \ell_3 (d_2 / d_3)(M_1^{(2)} / M_2^{(2)}) \} \tag{5.75a}$$

$$\gamma_{12}^{(2)} = \ell_2 \ell_3 (M_1^{(2)} / M_2^{(2)}) / (1-\ell) - \ell_2 \alpha_3 (d_3 / d_2) \tag{5.75b}$$

$$\mu_{11}^{(2)} = -\ell_2 \ell_3 C_2^{(2)} / (1-\ell) - \ell_3 \alpha_2 (d_2 / d_3) C_2^{(2)} + (M_3^{(2)} / M_2^{(2)})) \tag{5.76a}$$

$$\mu_{12}^{(2)} = \ell_2 \alpha_3 (d_3 / d_2) C_1^{(2)} + (1-\ell_2) a_3^{(2)} C_2^{(2)} + \ell_2 \ell_3 (M_3^{(2)} / M_2^{(2)}) / (1-\ell) \tag{5.76b}$$

and $W_2^{(2)}$ satisfies the following system:

$$\begin{cases} b_1^{(2)}(w_{22}^{(2)})^2 + b_2^{(2)}w_{22}^{(2)} + b_3^{(2)} = 0 & ; \\ w_{22}^{(2)} = -(N_2^{(2)} / N_1^{(2)})w_{22}^{(2)} - (N_3^{(2)} / N_1^{(2)}); \end{cases} \tag{5.77}$$

$$\begin{cases} v_{21}^{(2)} = A_1^{(2)} + \ell_2\ell_3(A_1^{(2)} - w_{21}^{(2)}) / (1-\ell) + (d_2 / d_3)\ell_3(1-\ell_2) \\ (A_2^{(2)} - w_{22}^{(2)}) / (1-\ell); \\ v_{22}^{(2)} = w_{22}^{(2)} + \ell_2\ell_3(w_{22}^{(2)} - A_2^{(2)}) / (1-\ell) + (d_3 / d_2)\ell_2(1-\ell_3) \\ (w_{21}^{(2)} - A_1^{(2)}) / (1-\ell); \end{cases} \tag{5.78}$$

where,

$$b_1^{(2)} = \frac{[((\gamma_{21}^{(2)})^2 - (N_2^{(2)} / N_1^{(2)})^2)c_2^+ - ((\eta_{21}^{(2)})^2 c_2^-)]}{2d_2(1-\ell_2)} + \frac{[((\gamma_{22}^{(2)})^2 c_3^- - (\eta_{22}^{(2)})^2 c_3^+)]}{2d_3(1-\ell_3)} \tag{5.79a}$$

$$b_2^{(2)} = \frac{(\gamma_{21}^{(2)}\mu_{21}^{(2)} - (N_2^{(2)}N_3^{(2)} / (N_1^{(2)})^2)c_2^+ - (\eta_{21}^{(2)})(v_{21}^{(2)} - \delta_2 d_2)c_2^-}{d_2(1-\ell_2)} + \frac{(\gamma_{22}^{(2)}(\mu_{22}^{(2)} - \delta_3 d_3) + \delta_3 d_3)c_3^- - \eta_{22}^{(2)}v_{22}^{(2)}c_2^-)]}{d_3(1-\ell_3)} \tag{5.79b}$$

$$b_3^{(2)} = \frac{[((\mu_{21}^{(2)})^2 - (N_3^{(2)} / N_1^{(2)})^2 - (I_1^{(2)})^2))c_2^+ - (v_{21}^{(2)} - \delta_2 d_2)^2 c_2^-]}{2d_2(1-\ell_2)} + \frac{\mu_{22}^{(2)}(\mu_{22}^{(2)} - 2\delta_3 d_3)^2 c_3^- - (v_{22}^{(2)})^2 - (I_2^{(2)})^2 c_3^+)]}{2d_3(1-\ell_3)} \tag{5.79c}$$

$$\eta_{21}^{(2)} = \alpha_2\left\{(d_2 / d_3)\ell_3 - (N_2^{(2)} / N_1^{(2)})(1-\ell_3)\right\}; \tag{5.80a}$$

$$\eta_{22}^{(2)} = \ell_2\alpha_3(d_3 / d_2)(N_2^{(2)} / N_1^{(2)}) - \ell_2\ell_3 / (1-\ell); \tag{5.80b}$$

$$v_{21}^{(2)} = -\ell_2\ell_3 C_1^{(2)} / (1-\ell) - \ell_3\alpha_2(d_2 / d_3)C_2^{(2)} - (1-\ell_3)\alpha_2(N_3^{(2)} / N_1^{(2)}) - \alpha_2(d_2 / d_3)\delta_3 d_3; \tag{5.81a}$$

$$v_{22}^{(2)} = \ell_2\alpha_3(d_3 / d_2)(C_1^{(2)} + (N_3^{(2)} / N_1^{(2)})) + (1-\ell_2)\alpha_3 C_2^{(2)} + \delta_3 d_3\ell_2 / (1-\ell); \tag{5.81b}$$

$$\gamma_{21}^{(2)} = \ell_2\ell_3(N_2^{(2)} / N_1^{(2)}) / (1-\ell) - \ell_3\alpha_2(d_2 / d_3); \tag{5.82a}$$

$$\gamma_{22}^{(2)} = \alpha_3\{(1-\ell) - \ell_2(d_3 / d_2)(N_2^{(2)} / N_1^{(2)})\}; \tag{5.82b}$$

$$\mu_{21}^{(2)} = (1-\ell_3)\alpha_2 A_1^{(2)} + \ell_3\alpha_2(d_2 / d_3)A_2^{(2)} + \ell_3\ell_2(N_3^{(2)} / N_1^{(2)}) / (1-\ell); \tag{5.83a}$$

$$\mu_{22}^{(2)} = -\ell_2\alpha_3(d_2 / d_3)(A_1^{(2)} + (N_3^{(2)} / N_1^{(2)})) - \ell_2\ell_3 A_2^{(2)} / (1-\ell). \tag{5.83b}$$

Notice that in (5.70), $w_{11}^{(2)}$ must be a real root of the quadratic such that $w_{11}^{(2)} \le C_1^{(2)}$. Similarly, $w_{22}^{(2)}$ in (5.77) must be a real root such that $w_{22}^{(2)} \le A_2^{(2)}$. If either quadratic in (5.70) or (5.77) has no root, then the corridor window does not exist in this case meaning that it is always optimal to reach the extracted schedule form toe their side (i.e. the side with the corridor window).

5.2.8 For Region R_3^u

Equations of boundaries $B_1^{(3)}$ *and* $B_2^{(3)}$:

$$B_1^{(3)} = M_1^{(3)}x_1 + M_2^{(3)}x_3 + M_3^{(3)} = 0; \tag{5.84}$$

$$M_1^{(3)} = c_1^- / (1-\ell_1) - \ell_1\ell_3 c_1^+ / (1-\ell_1)(1-\ell) - (d_1/d_3)(\ell_1/\ell_3)\alpha_3 c_3^-; \tag{5.85a}$$

$$M_2^{(3)} = -(d_1/d_3)\ell_3 c_1^+ / (1-\ell) - c_1^+ / \ell_3 - \alpha_3(1-\ell_3)c_3^- / \ell_3; \tag{5.85b}$$

$$M_3^{(3)} = -(M_1^{(3)}C_1^{(3)} + M_2^{(3)}C_2^{(3)}). \tag{5.85c}$$

and

$$B_2^{(3)} = N_1^{(3)}x_1 + N_2^{(3)}x_3 + N_3^{(3)} = 0; \tag{5.86}$$

$$N_1^{(3)} = -(d_3/d_1)\ell_1 c_3^+ / (1-\ell) - c_1^+ / \ell_1 - \alpha_1(1-\ell_3)c_1^- / \ell_1; \tag{5.87a}$$

$$N_2^{(3)} = c_3^- / (1-\ell_3) - \ell_1\ell_3 c_3^+ / (1-\ell_3)(1-\ell) - (d_3/d_1)(\ell_3/\ell_1)\alpha_3 c_1^-; \tag{5.87b}$$

$$N_3^{(3)} = -(N_1^{(3)}A_1^{(3)} + N_2^{(3)}A_2^{(3)}). \tag{5.88}$$

Corridor windows are:

Let

$$W_1^{(3)} = (w_{11}^{(3)}, w_{12}^{(3)}), V_1^{(3)} = (v_{11}^{(3)}, v_{12}^{(3)})$$

$$W_2^{(3)} = (w_{21}^{(3)}, w_{22}^{(2)}), V_2^{(2)} = (v_{21}^{(2)}, v_{22}^{(2)})$$

in *x-space*. $W_1^{(3)}$ satisfies the following system:

$$\begin{cases} a_1^{(3)}(w_{11}^{(3)})^2 + a_2^{(3)}w_{11}^{(3)} + a_3^{(3)} = 0 \qquad ; \\ w_{12}^{(3)} = -(M_1^{(3)}/M_2^{(3)})w_{11}^{(3)} - (M_3^{(3)}/M_2^{(3)}); \end{cases} \tag{5.89}$$

$$\begin{cases} v_{11}^{(3)} = w_{11}^{(3)} + \ell_1\ell_3(w_{11}^{(3)} - C_1^{(3)})/(1-\ell) + (d_1/d_3)\ell_3(1-\ell_1) \\ (w_{12}^{(3)} - C_2^{(3)})/(1-\ell); \\ v_{12}^{(3)} = C_2^{(3)} + \ell_1\ell_3(C_2^{(3)} - w_{12}^{(3)})/(1-\ell) + (d_3/d_1)\ell_1(1-\ell_3) \\ (C_1^{(3)} - w_{11}^{(3)})/(1-\ell); \end{cases} \tag{5.90}$$

where

$$a_1^{(3)} = \frac{((\gamma_{11}^{(3)})^2 - 1)c_1^- - (\eta_{11}^{(3)})^2 c_1^+}{2d_1(1-\ell_1)} + \frac{((\gamma_{12}^{(3)})^2 - (M_1^{(3)}/M_2^{(3)})^2)c_3^+ - (\eta_{12}^{(3)})^2 c_3^-}{2d_3(1-\ell_3)} \tag{5.91a}$$

$$a_2^{(3)} = \frac{(\gamma_{11}^{(3)}(\mu_{11}^{(3)} - \delta_1 d_1) + \delta_1 d_1)c_1^- - \eta_{11}^{(3)} v_{11}^{(3)} c_1^+}{d_1(1-\ell_1)} + \frac{(\gamma_{12}^{(3)}\mu_{12}^{(3)} - M_1^{(3)}M_3^{(3)}/(M_2^{(3)})^2)c_3^+ - \eta_{12}^{(3)}(v_{12}^{(3)} - \delta_3 d_3)c_3^-}{d_3(1-\ell_3)} \tag{5.91b}$$

$$a_3^{(3)} = \frac{\mu_{11}^{(3)}(\mu_{11}^{(3)} - 2\delta_1 d_1)c_1^- - ((v_{11}^{(3)})^2 - (I_2^{(3)})^2)c_1^+}{2d_1(1-\ell_1)} + \frac{((\mu_{12}^{(3)})^2 - (M_3^{(3)}/M_2^{(3)})^2 - (I_2^{(3)})^2)c_3^+ - (v_{12}^{(3)} - \delta_3 d_3)^2 c_3^-}{2d_3(1-\ell_3)} - k_3 \tag{5.91c}$$

$$\eta_{11}^{(3)} = \ell_3(d_1/d_3)(M_1^{(3)}/M_2^{(3)})\alpha_1 - \ell_1/(1-\ell) \tag{5.92a}$$

$$\eta_{12}^{(3)} = \alpha_3\{(d_3/d_1)\ell_1 - (M_1^{(3)}/M_2^{(3)})(1-\ell_1)\} \tag{5.92b}$$

$$v_{11}^{(3)} = \alpha_1(1-\ell_3)A_1^{(3)} + \alpha_1\ell_3(d_1/d_3)(A_2^{(3)} + (M_3^{(3)}/M_2^{(3)})) + \delta_1 d_1 \ell_3/(1-\ell) \tag{5.93a}$$

$$v_{12}^{(3)} = -\ell_1\alpha_3(d_3/d_1)A_1^{(3)} - \ell_1\ell_3 A_2^{(3)}/(1-\ell) - (1-\ell_3)\alpha_1(M_3^{(3)}/M_2^{(3)}) - \alpha_3(d_3/d_1)\delta_1 d_1 \tag{5.93b}$$

$$\gamma_{11}^{(3)} = \alpha_1\{(1-\ell_3) - \ell_3(d_1/d_3)(M_1^{(3)}/M_2^{(3)})\} \tag{5.94a}$$

$$\gamma_{12}^{(3)} = \ell_1\ell_3(M_1^{(3)}/M_2^{(3)})/(1-\ell) - \ell_1\alpha_3(d_3/d_1) \tag{5.94b}$$

$$\mu_{11}^{(3)} = -\ell_1\ell_3 C_2^{(3)}/(1-\ell) - \ell_3\alpha_1(d_1/d_3)C_2^{(3)} + (M_3^{(3)}/M_2^{(3)})) \tag{5.95a}$$

$$\mu_{12}^{(3)} = \ell_1\alpha_3(d_3/d_1)C_1^{(3)} + (1-\ell_1)a_3^{(3)}C_2^{(3)} + \ell_1\ell_3(M_3^{(3)}/M_2^{(3)})/(1-\ell) \tag{5.95b}$$

and $W_2^{(3)}$ satisfies the following system:

$$\begin{cases} b_1^{(3)}(w_{22}^{(3)})^2 + b_2^{(3)}w_{22}^{(3)} + b_3^{(3)} = 0 \quad ; \\ w_{22}^{(3)} = -(N_2^{(3)}/N_1^{(3)})w_{22}^{(3)} - (N_3^{(3)}/N_1^{(3)}); \end{cases} \tag{5.96}$$

$$\begin{cases} v_{21}^{(3)} = A_1^{(3)} + \ell_1\ell_3(A_1^{(3)} - w_{21}^{(3)})/(1-\ell) + (d_1/d_3)\ell_3(1-\ell_1) \\ (A_2^{(3)} - w_{22}^{(3)})/(1-\ell); \\ v_{22}^{(3)} = w_{22}^{(3)} + \ell_1\ell_3(w_{22}^{(3)} - A_2^{(3)})/(1-\ell) + (d_3/d_1)\ell_1(1-\ell_3) \\ (w_{21}^{(3)} - A_1^{(3)})/(1-\ell); \end{cases} \tag{5.97}$$

where,

$$b_1^{(3)} = \frac{[((\gamma_{21}^{(3)})^2 - (N_2^{(3)}/N_1^{(3)})^2)c_1^+ - ((\eta_{21}^{(3)})^2 c_1^-)]}{2d_1(1-\ell_1)} + \frac{[((\gamma_{22}^{(3)})^2 c_3^- - (\eta_{22}^{(3)})^2 c_3^+)]}{2d_3(1-\ell_3)} \tag{5.98a}$$

$$b_2^{(3)} = \frac{(\gamma_{21}^{(3)}\mu_{21}^{(3)} - (N_2^{(3)}N_3^{(3)} / (N_1^{(3)})^2)c_1^+ - (\eta_{21}^{(3)})(v_{21}^{(3)} - \delta_1 d_1)c_1^-}{d_1(1-\ell_1)} + \frac{(\gamma_{22}^{(3)}(\mu_{22}^{(3)} - \delta_3 d_3) + \delta_3 d_3)c_3^- - \eta_{22}^{(3)}v_{22}^{(3)}c_1^-)]}{d_3(1-\ell_3)} \quad (5.98b)$$

$$b_3^{(3)} = \frac{[((\mu_{21}^{(3)})^2 - (N_3^{(3)} / N_1^{(3)})^2 - (I_1^{(3)})^2))c_1^+ - (v_{21}^{(3)} - \delta_1 d_1)^2 c_1^-]}{2d_1(1-\ell_1)} + \frac{\mu_{22}^{(3)}(\mu_{22}^{(3)} - 2\delta_3 d_3)^2 c_3^- - (v_{22}^{(3)})^2 - (I_2^{(3)})^2 c_3^+)]}{2d_3(1-\ell_3)} \quad (5.99c)$$

$$\eta_{21}^{(3)} = \alpha_1\left\{(d_1/d_3)\ell_3 - (N_2^{(3)}/N_1^{(3)})(1-\ell_3)\right\}; \quad (6.00a)$$

$$\eta_{22}^{(3)} = \ell_1\alpha_3(d_3/d_1)(N_2^{(3)}/N_1^{(3)}) - \ell_1\ell_3/(1-\ell); \quad (6.00b)$$

$$v_{21}^{(3)} = -\ell_1\ell_3 C_1^{(3)}/(1-\ell) - \ell_3\alpha_1(d_1/d_3)C_2^{(3)} - (1-\ell_3)\alpha_1(N_3^{(3)}/N_1^{(3)}) - \alpha_1(d_1/d_3)\delta_3 d_3; \quad (6.01a)$$

$$v_{22}^{(3)} = \ell_1\alpha_3(d_3/d_1)(C_1^{(3)} + (N_3^{(3)}/N_1^{(3)})) + (1-\ell_1)\alpha_3 C_2^{(3)} + \delta_3 d_3 \ell_1/(1-\ell); \quad (6.01b)$$

$$\gamma_{21}^{(3)} = \ell_1\ell_3(N_2^{(3)}/N_1^{(3)})/(1-\ell) - \ell_3\alpha_1(d_1/d_3); \quad (6.02a)$$

$$\gamma_{22}^{(3)} = \alpha_3\{(1-\ell) - \ell_1(d_3/d_1)(N_2^{(3)}/N_1^{(3)})\}; \quad (6.02b)$$

$$\mu_{21}^{(3)} = (1-\ell_3)\alpha_1 A_1^{(3)} + \ell_3\alpha_1(d_1/d_3)A_2^{(3)} + \ell_3\ell_1(N_3^{(3)}/N_1^{(3)})/(1-\ell); \quad (6.03a)$$

$$\mu_{22}^{(3)} = -\ell_1\alpha_3(d_1/d_3)(A_1^{(3)} + (N_3^{(3)}/N_1^{(3)})) - \ell_1\ell_3 A_2^{(3)}/(1-\ell). \quad (6.03b)$$

Notice that in (5.89), $w_{11}^{(3)}$ must be a real root of the quadratic such that $w_{11}^{(3)} \le C_1^{(3)}$.

Similarly, $w_{22}^{(3)}$ in (5.96) must be a real root such that $w_{22}^{(3)} \le A_2^{(3)}$. If either quadratic in (5.89) or (5.96) has no root, then the corridor window does not exist in this case meaning that it is always optimal to reach the extracted schedule form toe their side (i.e. the side with the corridor window).

5.2.9 Optimal Trajectory in Region $R^u(R_1^u, R_2^u, and\ R_3^u)$

Based on Facts 1 and 2, the Optimal Trajectory emanating from a point in Region $R^u(R_1^u, R_2^u, and\ R_3^u)$ and leading to the extracted schedule in finite time can be obtained as follows: Given an initial surplus point in each of Region Region $R^u(R_1^u, R_2^u, and\ R_3^u)$; the first setup is chosen and the cost of the trajectory leading to the extracted schedule with two setup changes only is calculated. At this point; in each case, there is a $D_i - 2 - step$ trajectory, where *i* is the initial setup for product Type *i*.

Next, attempt was made to lower the cost of the current trajectory by introducing a setup change to the other product Type so that the extracted schedule is reached at the opposite side.

Therefore, if the cost can be reduced, then the extracted schedule is read at the opposite side.

Therefore, if the cost can be reduced, that the obtained new trajectory is a $D_i - 3 - step$ trajectory. By keeping the trying to reduce the cost of the current trajectory by introducing each time, a setup change before the extracted schedule is reached until the cost cannot be lowered anymore. At this point, there is an optimal $D_i - 3 - step$ trajectory emanating in each Region $R^u(R_1^u, R_2^u, and\ R_3^u)$ and reaching the extracted schedule in finite time. Similarly, the optimal $D_j - n - step$ (where $(j \neq i \neq l)$ trajectories are obtained starting with a setup to product Type *j* and *l* first in each case respectively. Thus, the optimal trajectory would be the one with the lowest cost.

Bai and Elhafsi (1996) and Onanaye (2006) with the numerical solutions of various examples, suggest that the above procedure be further simplified based on the following conjecture.

CONJECTURES

The initial setup of the optimal trajectory is given by $i \neq \arg\min_{i=1,2}\{C_i^2(x)\}$ is the cost of the $D_i - 2 - step$ trajectory staring with a setup change to product Type *i* and reaching the extracted schedule with two setup changes only. Similar conjecture for product type *j* and *l* with initial setup of the optimal trajectory given by

$j \neq \arg\min_{j=1,2}\{C_j^2(x)\}$ and

$l \neq \arg\min_{l=1,2}\{C_l^2(x)\}$ respectively.

Based on the above Conjectures, the optimal trajectory originating from $X = (a, b)$ in Region $R^u(R_1^u, R_2^u, and\ R_3^u)$ is obtained using the following procedures:

PROCEDURE

Module 1: $\mathrm{Re}\,gion\ R_1^u$

STEP 0: $X_1(0) := a^{(1)}, X_2(0) := b^{(1)}; k := 1;$

$IF\ (C_1^{(1)})^2 X^{(1)}(0)) \geq (C_2^{(1)})^2 (X^{(1)}(0))$ THEN

$X_1(k) := X_1(0) - \delta_2 d_1;$

$X_2(k) := X_2(0) - \delta_2 d_2;$

GOTO STEP 1;

ELSE

$X_1(k) := X_1(0) - \delta_1 d_1;$

$X_2(k) := X_2(0) - \delta_1 d_2;$

ENDIF;

STEP 1: $k := k + 1$

$X_1(k) := (d_1 \ell_2 M_3^{(1)} + X_2(k-1) M_2^{(1)} +$

$d_2(1 - \ell_2) X_1(k-1) M_2^{(1)}) / (d_2(1 - \ell_2) M_2^{(1)} - d_1 \ell_2 M_1^{(1)});$

$X_2(k) := -(M_1^{(1)} / M_2^{(1)}) X_1(k) - (M_3^{(1)} / M_2^{(1)});$

IF $X_1(k) \geq w_{11}^{(1)}$ THEN

$X_1(k) := \gamma_{11}^{(1)} X_1(k-1) + \mu_{11}^{(1)};$

$X_2(k) := \gamma_{12}^{(1)} X_2(k-1) + \mu_{12}^{(1)};$

$k := k + 1$

$X(k) := I^{(1)}$

GOTO STEP 3;

ELSE

GOTO STEP 2:

END IF;

STEP 2: $k := k + 1$

$X_2(k) := (d_2 \ell_1 (N_3^{(1)} + X_1(k-1) N_1^{(1)} +$

$d_1(1 - \ell_1) X_1(k-1) N_1^{(1)}) / (d_1(1 - \ell_1) N_1^{(1)} - d_2 \ell_1 N_2^{(1)});$

$X_1(k) := -(N_2^{(1)} / N_1^{(1)}) X_2(k) - (N_3^{(1)} / N_1^{(1)});$

IF $X_2(k) \geq w_{22}^{(1)}$ THEN

$X_1(k) := \gamma_{21}^{(1)} X_1(k-1) + \mu_{21}^{(1)};$

$X_2(k) := \gamma_{22}^{(1)} X_2(k-1) + \mu_{22}^{(1)};$

$k := k + 1$

$X_1(k) := X_1(k-1) - \delta_2 d_1;$

$X_2(k) := X_2(k-1) - \delta_2 d_2;$

$k := k + 1$

$X(k) := I^{(1)}$

GOTO STEP 3;

ELSE

GOTO STEP 1:

END IF;

STEP 3: OUTPUT: OPTIMAL TRAJECOTRY =

$\left\{X^{(1)}(0), X^{(1)}(1), \ldots., X^{(1)}(k-1), X^{(1)}(k)\right\};$

Module 2: $\text{Re}\,gion\ R_2^u$

STEP 0: $X_2(0) := a^{(2)}, X_3(0) := b^{(2)}; k := 1;$

$IF\ (C_1^{(2)})^2 X^{(2)}(0)) \geq (C_2^{(2)})^2 (X^{(2)}(0))$ THEN

$X_2(k) := X_2(0) - \delta_3 d_2;$

$X_3(k) := X_3(0) - \delta_3 d_3;$

GOTO STEP 1;

ELSE

$X_2(k) := X_2(0) - \delta_2 d_2;$

$X_3(k) := X_3(0) - \delta_2 d_3;$

ENDIF;

STEP 1: $k := k+1$

$X_2(k) := (d_2 \ell_3 M_3^{(2)} + X_3(k-1) M_2^{(2)} +$

$d_3(1-\ell_3) X_2(k-1) M_2^{(2)}) / (d_3(1-\ell_3) M_2^{(2)} - d_2 \ell_3 M_2^{(2)});$

$X_3(k) := -(M_1^{(2)} / M_2^{(2)}) X_2(k) - (M_3^{(2)} / M_2^{(2)});$

IF $X_2(k) \geq w_{11}^{(2)}$ THEN

$X_2(k) := \gamma_{11}^{(2)} X_2(k-1) + \mu_{11}^{(2)};$

$X_3(k) := \gamma_{12}^{(2)} X_3(k-1) + \mu_{12}^{(2)};$

$k := k+1$

$X(k) := I^{(2)}$

GOTO STEP 3;

ELSE

GOTO STEP 2:

END IF;

STEP 2: $k := k+1$

$X_3(k) := (d_3 \ell_2 (N_3^{(2)} + X_2(k-1) N_1^{(2)} +$

$d_2(1-\ell_2) X_2(k-1) N_1^{(2)}) / (d_2(1-\ell_2) N_1^{(2)} - d_3 \ell_2 N_2^{(2)});$

$X_2(k) := -(N_2^{(2)} / N_1^{(2)}) X_3(k) - (N_3^{(2)} / N_1^{(2)});$

IF $X_3(k) \geq w_{22}^{(2)}$ THEN

$X_2(k) := \gamma_{21}^{(2)} X_2(k-1) + \mu_{21}^{(2)};$

$X_3(k) := \gamma_{22}^{(2)} X_3(k-1) + \mu_{22}^{(2)};$

$k := k+1$

$X_2(k) := X_2(k-1) - \delta_3 d_2;$

$X_3(k) := X_3(k-1) - \delta_3 d_3;$

$k := k+1$

$X(k) := I^{(2)}$

GOTO STEP 3;

ELSE

GOTO STEP 1:

END IF;

STEP 3: OUTPUT: OPTIMAL TRAJECOTRY =

$\{X^{(2)}(0), X^{(2)}(1), ..., X^{(2)}(k-1), X^{(2)}(k)\};$

Module 3: $\text{Re}\,gion\ R_3^u$

STEP 0: $X_1(0) := a^{(3)}, X_3(0) := b^{(3)}; k := 1;$

$IF\ (C_1^{(3)})^2 X^{(3)}(0)) \geq (C_2^{(3)})^2 (X^{(3)}(0))$ THEN

$X_1(k) := X_1(0) - \delta_3 d_1;$

$X_3(k) := X_3(0) - \delta_3 d_3;$

GOTO STEP 1;

ELSE

$X_1(k) := X_1(0) - \delta_1 d_1;$

$X_3(k) := X_3(0) - \delta_1 d_3;$

ENDIF;

STEP 1: $k := k+1$

$X_1(k) := (d_1 \ell_2 M_3^{(3)} + X_2(k-1) M_2^{(3)} +$

$d_3(1-\ell_3) X_1(k-1) M_2^{(3)}) / (d_3(1-\ell_3) M_3^{(3)} - d_1 \ell_3 M_3^{(3)});$

$X_3(k) := -(M_1^{(3)} / M_2^{(3)}) X_1(k) - (M_3^{(3)} / M_2^{(3)});$

IF $X_1(k) \geq w_{11}^{(3)}$ THEN

$X_1(k) := \gamma_{11}^{(3)} X_1(k-1) + \mu_{11}^{(3)};$

$X_2(k) := \gamma_{12}^{(3)} X_2(k-1) + \mu_{12}^{(3)};$

$k := k+1$

$X(k) := I^{(3)}$

GOTO STEP 3;

ELSE

GOTO STEP 2:

END IF;

STEP 2: $k := k+1$

$X_3(k) := (d_3 \ell_1 (N_3^{(3)} + X_1(k-1) N_1^{(3)} +$

$d_1(1-\ell_1) X_1(k-1) N_1^{(3)}) / (d_1(1-\ell_1) N_1^{(3)} - d_3 \ell_1 N_2^{(3)});$

$X_1(k) := -(N_2^{(3)} / N_1^{(3)}) X_3(k) - (N_3^{(3)} / N_1^{(3)});$

IF $X_3(k) \geq w_{22}^{(3)}$ THEN

$X_1(k) := \gamma_{21}^{(3)} X_1(k-1) + \mu_{21}^{(3)};$

$X_3(k) := \gamma_{22}^{(3)} X_3(k-1) + \mu_{22}^{(3)};$

$k := k+1$

$X_1(k) := X_1(k-1) - \delta_3 d_1;$

$X_3(k) := X_3(k-1) - \delta_3 d_3;$

$k := k+1$

$X(k) := I^{(3)}$

GOTO STEP 3;

ELSE

GOTO STEP 1:

END IF;

STEP 3: OUTPUT: OPTIMAL TRAJECOTRY =

$\left\{ X^{(3)}(0), X^{(3)}(1), \ldots., X^{(3)}(k-1), X^{(3)}(k) \right\};$

5.3 n-Product Case

In order to apply the result we got both for two-product and three-product cases to higher cases, we need to extract a schedule from the cyclic schedule, to obtain the optimal solution of the transient case. In the case of four-product case, we divide the *x-space* into twelve mutually exclusive major regions as follows:

(R_1^u, R_1^o for(x_2,x_1)pair, R_2^u, R_2^o $for(x_2, x_3)$ pair, R_3^u, R_3^o $for(x_2, x_4)$ pair, R_4^u, R_4^o $for(x_1, x_3)$ $pair$, R_5^u, R_5^o $for(x_1, x_4)$ $pair$,and R_6^u, R_6^o $for(x_3, x_4)$ $pair$. We did this because, when production is switched from one product Type to another; say from product Type 1 to product Type 2, everything about production of product Type 3 and Type 4 is zero. Therefore, it is more convenient to use (x_2, x_1), (x_2, x_3), (x_2, x_4),…, (x_4, x_3) x – spaces. The Regions $R_1^u, R_1^o; R_2^u, R_2^o; ...; R_6^u, R_6^o$ are defined as in the case of three-product as shown in our earlier result and followed the same method as demonstrated above. In the case of five-product case, we divide the *x-space* into twenty mutually exclusive major regions as follows:

(R_1^u, R_1^o for(x_1,x_2)pair, R_2^u, R_2^o $for(x_2, x_3)$ pair, R_3^u, R_3^o $for(x_2, x_4)$ pair, R_4^u, R_4^o $for(x_1, x_5)$ $pair$, R_5^u, R_5^o $for(x_1, x_3)$ $pair$, R_6^u, R_6^o $for(x_1, x_4)$ $pair$, R_7^u, R_7^o $for(x_3, x_4)$ $pair$, R_8^u, R_8^o $for(x_3, x_5)$ $pair$, R_9^u, R_9^o $for(x_2, x_5)$ $pair$, and R_{10}^u, R_{10}^o $for(x_5, x_4)$ $pair$. We did this because, when production is switched from one product Type to another; say from product Type 1 to product Type 2, everything about production of product Type 3 ,Type 4 and Type 5 is zero. Therefore, it is more convenient to use (x_2, x_1), (x_2, x_3), (x_2, x_4),…, (x_5, x_4) x – spaces. The Regions $R_1^u, R_1^o; R_2^u, R_2^o; ...; R_{10}^u, R_{10}^o$ are defined as in the case of three-product as shown in our earlier result and followed the same method as demonstrated above.

It then follows from here that by mathematical induction, given n-product case, we would divide the *x-space* into $n(n-1)$ mutually exclusive major regions, where n is the number of products. We divide the *x-space* into $n(n-1)$ mutually exclusive major regions as follows:

(R_1^u, R_1^o for(x_1,x_n)pair, R_2^u, R_2^o $for(x_2, x_3)$ pair, R_3^u, R_3^o $for(x_2, x_4)$ pair, R_4^u, R_4^o $for(x_1, x_5)$ $pair$, R_5^u, R_5^o $for(x_1, x_3)$ $pair$, R_6^u, R_6^o $for(x_1, x_4)$ $pair$, R_7^u, R_7^o $for(x_3, x_4)$ $pair$, R_8^u, R_8^o $for(x_3, x_5)$ $pair$, R_9^u, R_9^o $for(x_2, x_5)$ $pair$,…, $R_{[n(n-1)/2]}^u, R_{[n(n-1)/2]}^o$ $for(x_n, x_{n-1})$ $pair$

We did this because, when production is switched from one product Type to another; say from product Type 1 to product Type 2, everything about other products say product Type 3,…, and Type n is zero. Therefore, it is more convenient to use (x_n, x_1), (x_3, x_2),…, (x_n, x_{n-1}) x – spaces.

The Regions $R_1^u, R_1^o; R_2^u, R_2^o; ...; R_{[n(n-1)/2]}^u, R_{[n(n-1)/2]}^o$ are defined as follows:

$$R_1^u = \begin{Bmatrix} (x_1, x_n) \mid d_n(1-\ell_n)(x_1 - A_1^{(1)}) + d_1\ell_2(x_n - A_2^{(1)}) < 0; \\ d_n\ell_1(x_1 - C_1^{(1)}) + d_1(1-\ell_1)(x_n - C_2^{(1)}) < 0 \end{Bmatrix}$$

$$R_1^o = \begin{Bmatrix} (x_1, x_n) \mid d_n(1-\ell_n)(x_1 - A_1^{(1)}) + d_1\ell_n(x_n - A_2^{(1)}) \geq 0; \\ d_n\ell_1(x_1 - C_1^{(1)}) + d_1(1-\ell_1)(x_n - C_2^{(1)}) \geq 0 \end{Bmatrix}$$

where $A_1^{(1)}, A_2^{(1)}, C_1^{(1)}\ and\ C_2^{(1)}$ are defined as

$$\begin{aligned}
A_1^{(1)} &= S_1 + (d_1 / d_n)\ell_n\alpha_1 Q_n - \alpha_1(1-\ell_n)(Q_1 - \delta d_1);\\
A_2^{(1)} &= s_n + \delta_n d_n + (d_n / d_1)\ell_1\alpha_n(Q_1 - \delta d_1) - Q_n\ell_1\ell_n / (1-\ell);\\
C_1^{(1)} &= s_1 + \delta_1 d_1 + (d_1 / d_n)\ell_n\alpha_1(Q_n - \delta d_n) - Q_1\ell_1\ell_n / (1-\ell);\\
C_2^{(1)} &= S_n + (d_n / d_1)\ell_1\alpha_n Q_1 - \alpha_n(1-\ell_1)(Q_n - \delta d_n)
\end{aligned}$$

$$R_2^u = \begin{Bmatrix} (x_2, x_3) \mid d_3(1-\ell_3)(x_2 - A_1^{(2)}) + d_2\ell_3(x_3 - A_2^{(2)}) < 0; \\ d_3\ell_2(x_2 - C_1^{(2)}) + d_2(1-\ell_2)(x_3 - C_2^{(2)}) < 0 \end{Bmatrix}$$

$$R_2^o = \begin{Bmatrix} (x_2, x_3) \mid d_3(1-\ell_3)(x_2 - A_1^{(2)}) + d_2\ell_3(x_3 - A_2^{(2)}) \geq 0; \\ d_3\ell_2(x_2 - C_1^{(2)}) + d_2(1-\ell_2)(x_3 - C_2^{(2)}) \geq 0 \end{Bmatrix}$$

where $A_1^{(2)}, A_2^{(2)}, C_1^{(2)}\ and\ C_2^{(2)}$ are defined as

$$\begin{aligned}
A_1^{(2)} &= S_2 + (d_2 / d_3)\ell_3\alpha_2 Q_3 - \alpha_2(1-\ell_3)(Q_2 - \delta d_2);\\
A_2^{(2)} &= s_3 + \delta_3 d_3 + (d_3 / d_2)\ell_2\alpha_3(Q_2 - \delta d_2) - Q_3\ell_2\ell_3 / (1-\ell);\\
C_1^{(2)} &= s_2 + \delta_2 d_2 + (d_2 / d_3)\ell_3\alpha_2(Q_3 - \delta d_2) - Q_2\ell_2\ell_3 / (1-\ell);\\
C_2^{(2)} &= S_3 + (d_3 / d_2)\ell_2\alpha_3 Q_2 - \alpha_3(1-\ell_2)(Q_3 - \delta d_3)
\end{aligned}$$

$$.\ R_{[n(n-1)/2]}^u = \begin{Bmatrix} (x_{n-1}, x_n) \mid d_n(1-\ell_n)(x_{n-1} - A_1^{(r)}) + d_{n-1}\ell_n(x_n - A_2^{(r)}) < 0; \\ d_n\ell_{n-1}(x_{n-1} - C_1^{(r)}) + d_{n-1}(1-\ell_{n-1})(x_n - C_2^{(r)}) < 0 \end{Bmatrix}$$

and

$$R_{[n(n-1)/2]}^o = \begin{Bmatrix} (x_{n-1}, x_n) \mid d_n(1-\ell_n)(x_{n-1} - A_1^{(r)}) + d_{n-1}\ell_n(x_n - A_2^{(r)}) \geq 0; \\ d_n\ell_{n-1}(x_{n-1} - C_1^{(r)}) + d_{n-1}(1-\ell_{n-1})(x_n - C_2^{(r)}) \geq 0 \end{Bmatrix}$$

where $A_1^{(r)}, A_2^{(r)}, C_1^{(r)}\ and\ C_2^{(r)}$ are defined as

$$A_1^{(r)} = S_{n-1} + (d_{n-1} / d_n)\ell_n\alpha_{n-1}Q_n - \alpha_{n-1}(1-\ell_n)(Q_{n-1} - \delta d_{n-1});$$
$$A_2^{(r)} = s_n + \delta_n d_n + (d_n / d_{n-1})\ell_{n-1}\alpha_n(Q_{n-1} - \delta d_{n-1}) - Q_n\ell_{n-1}\ell_n / (1-\ell);$$
$$C_1^{(r)} = s_{n-1} + \delta_{n-1}d_{n-1} + (d_{n-1} / d_n)\ell_n\alpha_{n-1}(Q_n - \delta d_{n-1}) - Q_{n-1}\ell_{n-1}\ell_n / (1-\ell);$$
$$C_2^{(r)} = S_n + (d_n / d_{n-1})\ell_{n-1}\alpha_n Q_{n-1} - \alpha_n(1-\ell_{n-1})(Q_n - \delta d_n)$$

where $r = n(n-1)/2$.

We now carry out the transient analysis in two parts for each *x – space* pairs. In the first part, the optimal transient solution for all initial surplus levels in Regions $R^o(R_1^o, R_2^o, ..., R_{[n(n-1)/2]}^o)$ was determined. In the second part, the optimal transient solution for initial surplus in Regions $R^u(R_1^u, R_2^u, ..., R_{[n(n-1)/2]}^u)$ is derived. Without loss of generality, we shall index the product Types such that product Type 1 is the product Type with the largest setup time and setup cost, followed by product type 2,…, product type *n* (i.e. $\delta_1 \geq \delta_2 \geq ...\delta_{n-1} \geq \delta_n \; and \; k_1 \geq k_2 \geq ...k_{n-1} \geq k_n.$).

5.3.1 Optimal Transient Solution in Region $R^o(R_1^o, R_2^o, ..., R_{[n(n-1)/2]}^o)$

We applied the same techniques used in Bai and Elhafsi (1996) and Onanaye (2006) to determine the optimal solution for initial surplus levels in Regions $R^o(R_1^o, R_2^o, ..., R_{[n(n-1)/2]}^o)$. Each of the regions is partitioned into $[n(n-1)/2]$ mutually exclusive regions,
$T^{(1)}, T_1^{(1)}, and\, T_2^{(1)} \; for\, R_1^o; T^{(2)}, T_1^{(2)}, \; and\, T_2^{(2)} \; for\, R_2^o; T^{(3)}, T_1^{(3)}, \; and\, T_2^{(3)} \; for\, R_3^o; ...; T^{(n)}, T_1^{(n)}, \; and$
$T_2^{(n)} \; for\, R_{[n(n-1)/2]}^o)$

Then regions $T^{(1)}, T^{(2)}, T^{(3)}, ..., and\, T^{(n)}$ are further partitioned into eight mutually exclusive sub-regions.

These sub-regions are as follows:

$T_{11}^{(1)}, T_{12}^{(1)}, T_{21}^{(1)}, T_{22}^{(1)}, U_{11}^{(1)}, U_{12}^{(1)}, U_{21}^{(1)} \, and\, U_{22}^{(1)}) \; for\; T^{(1)}$;
$T_{11}^{(2)}, T_{12}^{(2)}, T_{21}^{(2)}, T_{22}^{(2)}, U_{11}^{(2)}, U_{12}^{(2)}, U_{21}^{(2)} \, and\, U_{22}^{(2)}) \; for\; T^{(2)}$;

$T_{11}^{(3)}, T_{12}^{(3)}, T_{21}^{(3)}, T_{22}^{(3)}, U_{11}^{(3)}, U_{12}^{(3)}, U_{21}^{(3)} \, and\, U_{22}^{(3)}) \; for\; T^{(3)}$;

;…; $T_{11}^{(n)}, T_{12}^{(n)}, T_{21}^{(n)}, T_{22}^{(n)}, U_{11}^{(n)}, U_{12}^{(n)}, U_{21}^{(n)} \, and\, U_{22}^{(n)}) \; for\; T^{(n)}$

The regions are defined as follows:

5.3.2 Region R_1^0 and its sub regions

$$T_{11}^{(1)} = \left\{ \begin{array}{l} (x_1, x_n) \mid x_1 - \delta_1 d_1 \geq 0; -d_n(x_1 - (s_1 + \delta_1 d_1)) + \\ d_n(x_n - S_n) \geq 0 \end{array} \right\};$$

$$T_{12}^{(1)} = \left\{ \begin{array}{l} (x_1, x_n) \mid d_n(x_1 - (s_1 + \delta_1 d_1)) - d_1(x_n - S_n) > 0; -d_n(x_1 - E_1^{(1)}) + \\ d_1(x_n - E_2^{(1)}) \geq 0; d_n(1 - \ell_n)(x_1 - A_1^{(1)}) + d_1 \ell_n (x_n - A_2^{(1)}) \geq 0 \end{array} \right\};$$

$$T_{21}^{(1)} = \left\{ \begin{array}{l} (x_1, x_n) \mid -d_n(x_1 - S_1) + d_1(x_n - (s_n + \delta_n d_n)) \geq 0; -d_n(x_1 - E_1^{(1)}) \\ -d_1(x_n - E_2^{(1)}) > 0; d_n \ell_1 (x_1 - C_1^{(1)}) + d_1(1 - \ell_1)(x_n - C_2^{(1)}) \geq 0 \end{array} \right\};$$

$$T_{22}^{(1)} = \left\{ (x_1, x_n) \mid x_n - \delta_n d_n \geq 0; d_n(x_1 - S_n) - d_1(x_n - (s_n + \delta_n d_n) > 0 \right\};$$

$$U_{11}^{(1)} = \left\{ \begin{array}{l} (x_1, x_n) \mid x_1 - \delta_1 d_1 \geq 0; -d_n(x_1 - (s_1 + \delta_1 d_1)) + \\ d_n(x_n - S_n) \geq 0; -d_n(x_1 - t_{11}^{(1)}) + d_1(x_n - t_{12}^{(1)}) \geq 0; - \\ d_n(1 - \ell_n)(x_1 - A_1^{(1)}) - d_1 \ell_n (x_n - A_2^{(1)}) \geq 0 \end{array} \right\};$$

$$U_{12}^{(1)} = \left\{ \begin{array}{l} (x_1, x_n) \mid d_1 \ell_1 (x_1 - C_1^{(1)}); -d_n(1 - \ell_1) + d_1(x_n - C_2^{(1)}) > 0; - \\ d_n(x_1 - t_{11}^{(1)}) + d_1(x_n - t_{12}^{(1)}) \geq 0; -d_n(1 - \ell_n)(x_1 - A_1^{(1)}) - \\ d_1 \ell_n (x_n - A_2^{(1)}) \geq 0 \end{array} \right\};$$

$$U_{21}^{(1)} = \left\{ \begin{array}{l} (x_1, x_n) \mid d_n(x_1 - I_1^{(1)}) - d_1(x_n - I_2^{(1)}) > 0; d_n(1 - \ell_n)(x_1 - A_1^{(1)}) - d_1 \ell_n (x_n - A_2^{(1)}) \geq 0; - \\ d_n \ell_1 (x_1 - C_1^{(1)}) - d_1(1 - \ell_1)(x_n - C_2^{(1)}) > 0; -d_n(x_1 - t_{21}^{(1)}) + d_1(x_n - t_{22}^{(1)}) \geq 0 \end{array} \right\};$$

$$U_{22}^{(1)} = \left\{ \begin{array}{l} (x_1, x_n) \mid x_n - \delta_n d_n \geq 0; -d_n(x_1 - S_1) - d_1(x_n - (s_n + \delta_n d_n) > 0; -d_n \ell_1 (x_1 - C_1^{(1)}) \\ -d_1(1 - \ell_1)(x_n - C_2^{(1)}) > 0; -d_n(x_1 - t_{21}^{(1)}) + d_1(x_n - t_{22}^{(1)}) > 0 \end{array} \right\};$$

$$T_1^{(1)} = \left\{ \begin{array}{l} (x_1, x_n) \mid x_1 - \delta_1 d_1 > 0; d_n \ell_1 (x_1 - (s_1 + \delta_1 d_1)) + \\ d_1(1 - \ell_1)(x_n - S_n) \geq 0 \end{array} \right\}$$

$$T_2^{(1)} = \left\{ \begin{array}{l} (x_1, x_n) \mid -x_n + \delta_n d_n > 0; d_n(1 - \ell_n)(x_1 - S_1) + \\ d_1 \ell_n (x_n - (s_n + \delta_n d_n) \geq 0 \end{array} \right\}$$

where

$$t_1^{(1)} = (t_{11}^{(1)}, t_{12}^{(1)})^T$$

is the point in *x-space* given by:

$t_{11}^{(1)} = \delta_1 d_1$

$t_{11}^{(1)} = C_2^{(1)} + (C_1^{(1)} - \delta_1 d_1)\delta_n \ell_1 / d_1(1 - \ell_1).$

$t_2^{(1)} = (t_{21}^{(1)}, t_{22}^{(1)})^T$

is the point in *x-space* given by:

$t_{21}^{(1)} = A_1^{(1)} + (A_2^{(1)} - \delta_n d_n)\delta_1 \ell_n / d_n(1 - \ell_n).$

$t_{22}^{(1)} = \delta_2 d_2 .$

and

$E^{(1)} = (E_1^{(1)}, E_2^{(1)})^T$

is the point in *x-space* given by:

$E^{(1)} = (-\theta_2^{(1)} + \sqrt{(\theta_2^{(1)})^2 + \theta_1^{(1)}\theta_3^{(1)}}) / \theta_1^{(1)}$

and

$E_2^{(1)} = \xi_1^{(1)}(-\theta_2^{(1)} + \sqrt{(\theta_2^{(1)})^2 + \theta_1^{(1)}\theta_3^{(1)}}) / \theta_1^{(1)}) + \xi_1^{(1)}$

where

$\theta_1^{(1)} = (c_1^+ / d_1)((\xi_1^{(1)})^2 - (\xi_3^{(1)})^2) + (c_n^+ / d_n)(1 - (\xi_5^{(1)})^2);$

$\theta_2^{(1)} = (c_1^+ / d_1)(\xi_1^{(1)}\xi_2^{(1)} - \xi_3^{(1)}\xi_4^{(1)}) - (c_n^+ / d_n)\xi_5^{(1)}\xi_6^{(1)};$

$\theta_3^{(1)} = 2(k_1 - k_n) + (c_1^+ / d_1)[(\xi_4^{(1)})^2 - (\xi_2^{(1)})^2] + (c_n^+ / d_n)(\xi_6^{(1)})^2;$

$\xi_1^{(1)} = -(d_1 / d_n)(1 - \ell_1) / \ell_1;$

$\xi_2^{(1)} = A_1^{(1)} + A_2^{(1)}(d_1 / d_n)(1 - \ell_1) / \ell_1;$

$\xi_3^{(1)} = -(d_1 / d_n)\ell_n / \ell_1;$

$\xi_4^{(1)} = \ell_n A_1^{(1)} + (\ell_n / \ell_1)(1 - \ell_1)A_2^{(1)} + (1 - \ell_n)C_1^{(1)} + \ell_n(d_1 / d_n)C_2^{(1)};$

$\xi_5^{(1)} = (1 - \ell_n) / \ell_1;$

$$\xi_6^{(1)} = \ell_n C_2^{(1)} - A_2^{(1)}(1-\ell_n)(1-\ell_1)/\ell_1 + (d_n/d_1)(1-\ell_n)(C_2^{(1)} - A_1^{(1)};$$

5.3.3 Region R_2^0 and its sub regions

$$T_{11}^{(2)} = \begin{Bmatrix} (x_2,x_3) \mid x_2 - \delta_2 d_2 \geq 0; -d_3(x_2-(s_2+\delta_2 d_2)) + \\ d_3(x_3 - S_3) \geq 0 \end{Bmatrix};$$

$$T_{12}^{(2)} = \begin{Bmatrix} (x_2,x_3) \mid d_3(x_2-(s_2+\delta_2 d_2)) - d_2(x_3-S_3) > 0; -d_3(x_2 - E_1^{(2)}) + \\ d_3(x_3 - E_2^{(2)}) \geq 0; d_3(1-\ell_3)(x_2 - A_1^{(2)}) + d_2\ell_3(x_3 - A_2^{(2)}) \geq 0 \end{Bmatrix};$$

$$T_{21}^{(2)} = \begin{Bmatrix} (x_2,x_3) \mid -d_3(x_2 - S_2) + d_2(x_3-(s_3+\delta_3 d_3)) \geq 0; -d_3(x_2 - E_1^{(2)}) \\ -d_2(x_3 - E_2^{(2)}) > 0; d_3\ell_2(x_2 - C_1^{(2)}) + d_2(1-\ell_2)(x_3 - C_2^{(2)}) \geq 0 \end{Bmatrix};$$

$$T_{22}^{(2)} = \{(x_2,x_3) \mid x_3 - \delta_3 d_3 \geq 0; d_3(x_2 - S_3) - d_2(x_3-(s_3+\delta_3 d_3) > 0\};$$

$$U_{11}^{(2)} = \begin{Bmatrix} (x_2,x_3) \mid x_2 - \delta_2 d_2 \geq 0; -d_3(x_2-(s_2+\delta_2 d_2)) + \\ d_3(x_3 - S_3) \geq 0; -d_3(x_2 - t_{11}^{(2)}) + d_1(x_2 - t_{12}^{(2)}) \geq 0; - \\ d_3(1-\ell_3)(x_2 - A_1^{(2)}) - d_2\ell_3(x_3 - A_2^{(2)}) \geq 0 \end{Bmatrix};$$

$$U_{12}^{(2)} = \begin{Bmatrix} (x_2,x_3) \mid d_2\ell_2(x_2 - C_1^{(2)}); -d_3(1-\ell_2) + d_2(x_3 - C_2^{(2)}) > 0; - \\ d_3(x_2 - t_{11}^{(2)}) + d_2(x_3 - t_{12}^{(2)}) \geq 0; -d_3(1-\ell_3)(x_2 - A_1^{(2)}) - \\ d_2\ell_3(x_3 - A_2^{(2)}) \geq 0 \end{Bmatrix};$$

$$U_{21}^{(2)} = \begin{Bmatrix} (x_2,x_3) \mid d_3(x_2 - I_1^{(2)}) - d_2(x_3 - I_2^{(2)}) > 0; d_3(1-\ell_3)(x_2 - A_1^{(2)}) - d_1\ell_2(x_2 - A_2^{(2)}) \geq 0; - \\ d_3\ell_2(x_2 - C_1^{(2)}) - d_2(1-\ell_2)(x_3 - C_2^{(2)}) > 0; -d_3(x_2 - t_{21}^{(2)}) + d_2(x_3 - t_{22}^{(2)}) \geq 0 \end{Bmatrix};$$

$$U_{22}^{(2)} = \begin{Bmatrix} (x_2,x_3) \mid x_3 - \delta_3 d_3 \geq 0; -d_3(x_2 - S_2) - d_2(x_3-(s_3+\delta_3 d_3) > 0; -d_3\ell_2(x_2 - C_1^{(2)}) \\ -d_2(1-\ell_2)(x_3 - C_2^{(2)}) > 0; -d_3(x_2 - t_{21}^{(2)}) + d_2(x_3 - t_{22}^{(2)}) > 0 \end{Bmatrix};$$

$$T_1^{(2)} = \begin{Bmatrix} (x_2,x_3) \mid x_2 - \delta_2 d_2 > 0; d_3\ell_2(x_2-(s_2+\delta_2 d_2)) + \\ d_2(1-\ell_2)(x_3 - S_3) \geq 0 \end{Bmatrix};$$

$$T_2^{(2)} = \begin{Bmatrix} (x_2,x_3) \mid -x_3 + \delta_3 d_3 > 0; d_3(1-\ell_3)(x_2 - S_2) + \\ d_2\ell_3(x_3-(s_3+\delta_3 d_3) \geq 0 \end{Bmatrix};$$

where

$$t_1^{(2)} = (t_{11}^{(2)}, t_{12}^{(2)})^T$$

is the point in *x-space* given by:

$t_{11}^{(2)} = \delta_2 d_2$

$t_{11}^{(2)} = C_2^{(2)} + (C_1^{(2)} - \delta_2 d_2)\delta_3 \ell_2 / d_2(1-\ell_2).$

$t_2^{(2)} = (t_{21}^{(2)}, t_{22}^{(2)})^T$

is the point in *x-space* given by:

$t_{21}^{(2)} = A_1^{(2)} + (A_2^{(2)} - \delta_3 d_3)\delta_2 \ell_3 / d_3(1-\ell_3).$

$t_{22}^{(3)} = \delta_3 d_3.$

and

$E^{(2)} = (E_1^{(2)}, E_2^{(2)})^T$

is the point in *x-space* given by:

$E^{(2)} = (-\theta_2^{(2)} + \sqrt{(\theta_2^{(2)})^2 + \theta_1^{(2)}\theta_3^{(2)}}) / \theta_1^{(2)}$

and

$E_2^{(2)} = \xi_1^{(2)}(-\theta_2^{(2)} + \sqrt{(\theta_2^{(2)})^2 + \theta_1^{(2)}\theta_3^{(2)}}) / \theta_1^{(2)}) + \xi_1^{(2)}$

where

$\theta_1^{(2)} = (c_2^+ / d_2)((\xi_1^{(2)})^2 - (\xi_3^{(2)})^2 + (c_3^+ / d_3)(1-(\xi_5^{(2)})^2);$

$\theta_2^{(2)} = (c_2^+ / d_2)((\xi_1^{(2)}\xi_1^{(2)} - \xi_3^{(2)}\xi_4^{(2)}(c_3^+ / d_3)\xi_5^{(2)}\xi_6^{(2)});$

$\theta_3^{(2)} = 2(k_2 - k_3) + (c_2^+ / d_2)((\xi_4^{(2)})^2 - (\xi_2^{(2)})^2 + (c_3^+ / d_3)((\xi_6^{(2)})^2);$

$\xi_1^{(2)} = -(d_2 / d_3)(1-\ell_2) / \ell_2;$

$\xi_2^{(2)} = A_1^{(2)} + A_2^{(2)}(d_2 / d_3)(1-\ell_2) / \ell_2;$

$\xi_3^{(2)} = -(d_2 / d_3)\ell_3 / \ell_2;$

$\xi_4^{(2)} = \ell_2 A_1^{(2)} - (\ell_3 / \ell_2)(1-\ell_2)A_2^{(2)} + (1-\ell_3)C_1^{(2)} + \ell_3(d_2 / d_3)C_2^{(2)};$

$\xi_1^{(2)} = (1-\ell_3)/\ell_2;$

$\xi_6^{(2)} = \ell_3 C_2^{(2)} - A_2^{(2)}(1-\ell_3)(1-\ell_2)/\ell_2 + (d_3/d_2)(1-\ell_3)(C_1^{(2)} - A_1^{(2)});$

We shall continue likewise to compute for other regions until we get to Region $R^o_{[n(n-1)/2]}$ and we define it as follows:

5.3.4 Region $R^o_{[n(n-1)/2]}$ and its sub regions

$$T_{11}^{(r)} = \left\{\begin{matrix}(x_{n-1}, x_n) \mid x_{n-1} - \delta_{n-1}d_{n-1} \geq 0; -d_n(x_{n-1} - (s_{n-1} + \delta_{n-1}d_{n-1})) + \\ d_n(x_n - S_n) \geq 0\end{matrix}\right\};$$

where $r = n(n-1)/2$

$$T_{12}^{(r)} = \left\{\begin{matrix}(x_{n-1}, x_n) \mid d_n(x_{n-1} - (s_{n-1} + \delta_{n-1}d_{n-1})) - d_{n-1}(x_n - S_n) > 0; -d_n(x_{n-1} - E_1^{(r)}) + \\ d_{n-1}(x_n - E_2^{(r)}) \geq 0; d_n(1-\ell_n)(x_{n-1} - A_1^{(r)}) + d_{n-1}\ell_n(x_n - A_2^{(r)}) \geq 0\end{matrix}\right\};$$

$$T_{21}^{(r)} = \left\{\begin{matrix}(x_{n-1}, x_n) \mid -d_n(x_{n-1} - S_{n-1}) + d_{n-1}(x_n - (s_n + \delta_n d_n)) \geq 0; -d_n(x_{n-1} - E_1^{(1)}) \\ -d_{n-1}(x_n - E_2^{(r)}) > 0; d_n\ell_{n-1}(x_{n-1} - C_1^{(r)}) + d_{n-1}(1-\ell_{n-1})(x_n - C_2^{(r)}) \geq 0\end{matrix}\right\};$$

$$T_{22}^{(r)} = \left\{(x_{n-1}, x_n) \mid x_n - \delta_n d_n \geq 0; d_n(x_{n-1} - S_n) - d_{n-1}(x_n - (s_n + \delta_n d_n) > 0\right\};$$

$$U_{11}^{(r)} = \left\{\begin{matrix}(x_{n-1}, x_n) \mid x_{n-1} - \delta_{n-1}d_{n-1} \geq 0; -d_n(x_{n-1} - (s_{n-1} + \delta_{n-1}d_{n-1})) + \\ d_n(x_n - S_n) \geq 0; -d_n(x_{n-1} - t_{11}^{(r)}) + d_{n-1}(x_n - t_{12}^{(r)}) \geq 0; - \\ d_n(1-\ell_n)(x_{n-1} - A_1^{(r)}) - d_{n-1}\ell_n(x_n - A_2^{(r)}) \geq 0\end{matrix}\right\};$$

$$U_{12}^{(r)} = \left\{\begin{matrix}(x_{n-1}, x_n) \mid d_{n-1}\ell_{n-1}(x_{n-1} - C_1^{(r)}); -d_n(1-\ell_{n-1}) + d_{n-1}(x_n - C_2^{(r)}) > 0; - \\ d_n(x_{n-1} - t_{11}^{(r)}) + d_{n-1}(x_n - t_{12}^{(r)}) \geq 0; -d_n(1-\ell_n)(x_{n-1} - A_1^{(r)}) - \\ d_{n-1}\ell_n(x_n - A_2^{(r)}) \geq 0\end{matrix}\right\};$$

$$U_{21}^{(r)} = \left\{\begin{matrix}(x_{n-1}, x_n) \mid d_n(x_{n-1} - I_1^{(r)}) - d_{n-1}(x_n - I_2^{(r)}) > 0; d_n(1-\ell_n)(x_{n-1} - A_1^{(r)}) - d_{n-1}\ell_n(x_n - A_2^{(r)}) \geq 0; - \\ d_n\ell_{n-1}(x_{n-1} - C_1^{(r)}) - d_{n-1}(1-\ell_{n-1})(x_n - C_2^{(r)}) > 0; -d_n(x_{n-1} - t_{21}^{(r)}) + d_{n-1}(x_n - t_{22}^{(r)}) \geq 0\end{matrix}\right\};$$

$$U_{22}^{(r)} = \left\{\begin{matrix}(x_{n-1}, x_n) \mid x_n - \delta_n d_n \geq 0; -d_n(x_{n-1} - S_{n-1}) - d_{n-1}(x_n - (s_n + \delta_n d_n) > 0; -d_n\ell_{n-1}(x_{n-1} - C_1^{(r)}) \\ -d_{n-1}(1-\ell_{n-1})(x_n - C_2^{(r)}) > 0; -d_n(x_{n-1} - t_{21}^{(r)}) + d_{n-1}(x_n - t_{22}^{(r)}) > 0\end{matrix}\right\};$$

$$T_1^{(r)} = \left\{\begin{matrix}(x_{n-1}, x_n) \mid x_{n-1} - \delta_{n-1}d_{n-1} > 0; d_n\ell_{n-1}(x_{n-1} - (s_{n-1} + \delta_{n-1}d_{n-1})) + \\ d_{n-1}(1-\ell_{n-1})(x_n - S_n) \geq 0\end{matrix}\right\};$$

$$T_2^{(r)} = \begin{Bmatrix} (x_{n-1}, x_n) \mid -x_n + \delta_n d_n > 0; d_n(1-\ell_n)(x_{n-1} - S_{n-1}) + \\ d_{n-1}\ell_n(x_n - (s_n + \delta_n d_n) \geq 0 \end{Bmatrix};$$

where

$$t_1^{(r)} = (t_{11}^{(r)}, t_{12}^{(r)})^T$$

is the point in *x-space* given by:

$$t_{11}^{(r)} = \delta_{n-1} d_{n-1}$$

$$t_{11}^{(r)} = C_2^{(r)} + (C_1^{(r)} - \delta_{n-1} d_{n-1})\delta_n \ell_{n-1} / d_{n-1}(1 - \ell_{n-1}).$$

$$t_2^{(r)} = (t_{21}^{(r)}, t_{22}^{(r)})^T$$

is the point in *x-space* given by:

$$t_{21}^{(r)} = A_1^{(r)} + (A_2^{(r)} - \delta_n d_n)\delta_{n-1} \ell_n / d_n(1 - \ell_n).$$

$$t_{22}^{(r)} = \delta_n d_n.$$

and

$$E^{(r)} = (E_1^{(r)}, E_2^{(r)})^T$$

is the point in *x-space* given by:

$$E_1^{(r)} = (-\theta_2^{(r)} + \sqrt{\left[((\theta_2^{(r)})^2 + \theta_1^{(r)}\theta_3^{(r)}) / \theta_1^{(r)}\right]}$$

and

$$E_2^{(r)} = \xi_1^{(r)}(-\theta_2^{(r)} + \sqrt{\left[((\theta_2^{(r)})^2 + \theta_1^{(r)}\theta_3^{(r)})\right]} / (\theta_1^{(r)}) + \xi_2^{(r)}$$

where

$$\theta_1^{(r)} = (c_{n-1}^+ / d_{n-1})((\xi_1^{(r)})^2 - (\xi_3^{(r)})^2) + (c_n^+ / d_n)(1 - (\xi_5^{(r)})^2);$$

$$\theta_2^{(r)} = (c_{n-1}^+ / d_{n-1})(\xi_1^{(r)}\xi_2^{(r)} - \xi_3^{(r)}\xi_4^{(r)}) - (c_n^+ / d_n)\xi_5^{(r)}\xi_6^{(r)};;$$

$$\theta_3^{(r)} = 2(k_{n-1} - k_n) + (c_{n-1}^+ / d_{n-1})[(\xi_4^{(r)})^2 - (\xi_2^{(r)})^2] + (c_n^+ / d_n)(\xi_6^{(r)})^2;$$

$$\xi_1^{(r)} = -(d_{n-1}/d_n)(1-\ell_{n-1})/\ell_{n-1};$$

$$\xi_2^{(r)} = A_1^{(r)} + A_2^{(r)}(d_{n-1}/d_n)(1-\ell_{n-1})/\ell_{n-1};$$

$$\xi_3^{(r)} = -(d_{n-1}/d_n)\ell_n/\ell_{n-1};$$

$$\xi_4^{(r)} = \ell_n A_1^{(r)} + (\ell_n/\ell_{n-1})(1-\ell_{n-1})A_2^{(r)} + (1-\ell_n)C_1^{(r)} + \ell_n(d_{n-1}/d_n)C_2^{(r)};$$

$$\xi_5^{(r)} = (1-\ell_n)/\ell_{n-1};$$

$$\xi_6^{(r)} = \ell_n C_2^{(r)} - A_2^{(r)}(1-\ell_n)(1-\ell_{n-1})/\ell_{n-1} + (d_n/d_{n-1})(1-\ell_n)(C_2^{(r)} - A_1^{(r)}$$

5.3.5 Optimal Transient Solution in Region $R^u(R_1^u, R_2^u, ..., R_{[n(n-1)/2]}^u)$

The optimal solution for initial surplus levels in Region $R^u(R_1^u, R_2^u, ..., R_{[n(n-1)/2]}^u)$ is not obvious and is more involved mathematically. We established the following facts and definitions, based on the results established in Bai and Elhafsi (1996) and Onanaye (2006).

FACT 1: For initial surplus levels in Region $R^u(R_1^u, R_2^u, ..., R_{[n(n-1)/2]}^u)$, the optimal way to move (advance) towards the cyclic schedule is by producing at maximum machine speed whenever it is possible.

That is, $U^* = (U_1, 0, 0, ..., 0)\, or\, (0, U_2, 0, 0, ..., 0), or, ..., (0, 0, 0, ..., U_n)$ if the machine is producing and $U^* = (0, 0, 0, ..., 0)$ if the machine is undergoing a setup change.

Therefore, given the current setup state, the direction of the surplus trajectory in $R^u(R_1^u, R_2^u, ..., R_{[n(n-1)/2]}^u)$ can be known (or is known).

DEFINITION 1

A trajectory is said to be following Direction D_i, if it moves parallel to line L_i in the direction of increasing x_i. That is, the machine is producing product Type (i=1,2,3,…,n) at maximum speed.

DEFINITION 2

Let a $D_i - n - step\ trajectory\,(i = 1,2,3,...,n; n > 1)$, a trajectory that performs alternately m_i setup change –production runs of product Type j and m_l setup change-production runs of product Type $l(j \neq l \neq i \neq,...,z)$ with the initial setup change to product Type i and the case segment touching the cyclic schedule at any boundary. If n is even then $m_i = m_j = m_l = ... = m_z = n/N$. Where N is the number of product Types that the machine can produce.

If n is odd, then

$m_i = (n + 1)/N,$

$m_j = (n - 1)/N,$

.

.

.

and

$m_z = (n - 1)/N.$

FACT 2:

To reach the cyclic schedule in finite time, starting with initial surplus levels in Region $R^u(R_1^u, R_2^u, ..., R_{[n(n-1)/2]}^u)$ the trajectory leading to the cyclic schedule must touch the boundary $L_{i,j,l,...,z}(i,j,l,...,z = 1,2,3,...,n\ and\ i \neq j \neq l \neq ... \neq z)$ of Region $R^u(R_1^u, R_2^u, ..., R_{[n(n-1)/2]}^u)$, just before switching to product Type j or to product Type l *…or to product Type z,* and reaching the cyclic schedule.

In Elhafsi and Bai (1996b) and Onanaye (2006), it has been established that the optimal switching policy is a special corridor policy with two windows. In the case where production at the demand rates is possible in the cyclic schedule, this result is still valid. Its corridor windows defined by the pairs of points $(W_1^{(1)}, V_1^{(1)}; W_2^{(2)}, V_2^{(2)}; W_3^{(3)}, V_3^{(3)}; ...; W_r^{(r)}, V_r^{(r)})$; for each region.

Using the technique developed in Elhafsi and Bai (1996b) and Onanaye (2006), the corridor boundaries and windows are determined as follows:

5.3.6 For Region R_1^u

Equations of boundaries $B_1^{(1)}$ *and* $B_2^{(1)}$;

$$B_1^{(1)} = M_1^{(1)} x_1 + M_2^{(1)} x_n + M_3^{(1)} = 0; \qquad (6.04)$$

$$M_1^{(1)} = c_1^- / (1-\ell_1) - \ell_1 \ell_n c_1^+ / (1-\ell_1)(1-\ell) - (d_1 / d_n)(\ell_1 / \ell_n) \alpha_n c_n^- \qquad (6.05a)$$

$$M_2^{(1)} = -(d_1 / d_n) \ell_n c_1^+ / (1-\ell) - c_n^+ / \ell_n - \alpha_n (1-\ell_n) c_n^- / \ell_n \qquad (6.05b)$$

$$M_3^{(1)} = -(M_1^{(1)} C_1^{(1)} + M_2^{(1)} C_2^{(1)}) \qquad (6.05c)$$

and

$$B_2^{(1)} = N_1^{(1)} x_1 + N_2^{(1)} x_n + N_3^{(1)} = 0; \qquad (6.06)$$

$$N_1^{(1)} = -(d_n / d_1) \ell_1 c_n^+ / (1-\ell) - c_1^+ / \ell_1 - \alpha_1 (1-\ell_n) c_1^- / \ell_1; \qquad (6.05a)$$

$$N_2^{(1)} = c_n^- (1-\ell_n) - \ell_1 \ell_n c_n^+ / (1-\ell_n)(1-\ell) - (d_n / d_1)(\ell_n / \ell_1) \alpha_1 c_1^- / \ell_1; \qquad (6.05b)$$

$$N_3^{(1)} = -(N_1^{(1)} A_1^{(1)} + N_2^{(1)} A_2^{(1)}) \qquad (6.05c)$$

Corridor windows are:

Let

$$W_1^{(1)} = (w_{11}^{(1)}, w_{12}^{(1)}), V_1^{(1)} = (v_{11}^{(1)}, v_{12}^{(1)})$$

$$W_2^{(1)} = (w_{21}^{(1)}, w_{22}^{(1)}), V_2^{(1)} = (v_{21}^{(1)}, v_{22}^{(1)})$$

in *x-space*. $W_1^{(1)}$ satisfies the following system:

$$\begin{cases} a_1^{(1)} (w_{11}^{(1)})^2 + a_2^{(1)} w_{11}^{(1)} + a_3^{(1)} = 0 \quad ; \\ w_{12}^{(1)} = -(M_1^{(1)} / M_2^{(1)}) w_{11}^{(1)} - (M_3^{(1)} / M_2^{(1)}); \end{cases} \qquad (6.06)$$

$$\begin{cases} v_{11}^{(1)} = w_{11}^{(1)} + \ell_1 \ell_n (w_{11}^{(1)} - C_1^{(1)}) / (1-\ell) + (d_1 / d_n) \ell_n (1-\ell_1) \\ (w_{12}^{(1)} - C_2^{(1)}) / (1-\ell); \\ v_{12}^{(1)} = C_2^{(1)} + \ell_1 \ell_n (C_2^{(1)} - w_{12}^{(1)}) / (1-\ell) + (d_n / d_1) \ell_1 (1-\ell_n) \\ (C_1^{(1)} - w_{11}^{(1)}) / (1-\ell); \end{cases} \qquad (6.07)$$

where

$$a_1^{(1)} = \frac{((\gamma_{11}^{(1)})^2 - 1)c_1^- - (\eta_{11}^{(1)})^2 c_1^+}{2d_1(1-\ell_1)} + \frac{((\gamma_{12}^{(1)})^2 - (M_1^{(1)} / M_2^{(1)})^2)c_n^+ - (\eta_{12}^{(1)})^2 c_n^-}{2d_n(1-\ell_n)} \tag{6.08a}$$

$$a_2^{(1)} = \frac{(\gamma_{11}^{(1)}(\mu_{11}^{(1)} - \delta_1 d_1) + \delta_1 d_1)c_1^- - \eta_{11}^{(1)} v_{11}^{(1)} c_1^+}{d_1(1-\ell_1)} + \frac{(\gamma_{12}^{(1)}\mu_{12}^{(1)} - M_1^{(1)} M_3^{(1)} / (M_2^{(1)})^2)c_n^+ - \eta_{12}^{(1)}(v_{12}^{(1)} - \delta_n d_n)c_n^-}{d_n(1-\ell_n)} \tag{6.08b}$$

$$a_3^{(1)} = \frac{\mu_{11}^{(1)}(\mu_{11}^{(1)} - 2\delta_1 d_1)c_1^- - ((v_{11}^{(1)})^2 - (I_2^{(1)})^2)c_1^+}{2d_1(1-\ell_1)} + \frac{((\mu_{12}^{(1)})^2 - (M_3^{(1)} / M_2^{(1)})^2 - (I_2^{(1)})^2)c_n^+ - (v_{12}^{(1)} - \delta_n d_n)^2 c_n^-}{2d_n(1-\ell_n)} - k_n \tag{6.08c}$$

$$\eta_{11}^{(1)} = \ell_n (d_1 / d_n)(M_1^{(1)} / M_2^{(1)})\alpha_1 - \ell_1 / (1-\ell) \tag{6.09a}$$

$$\eta_{12}^{(1)} = \alpha_n \{(d_n / d_1)\ell_1 - (M_1^{(1)} / M_2^{(1)})(1-\ell_1)\} \tag{6.09b}$$

$$v_{11}^{(1)} = \alpha_1 (1-\ell_n) A_1^{(1)} + \alpha_1 \ell_n (d_1 / d_n)(A_2^{(1)} + (M_3^{(1)} / M_2^{(1)})) + \delta_1 d_1 \ell_n / (1-\ell) \tag{6.10a}$$

$$v_{12}^{(1)} = -\ell_1 \alpha_n (d_n / d_1) A_1^{(1)} - \ell_1 \ell_n A_2^{(1)} / (1-\ell) - (1-\ell_n)\alpha_1 (M_3^{(1)} / M_2^{(1)}) - \alpha_n (d_n / d_1)\delta_1 d_1 \tag{6.10b}$$

$$\gamma_{11}^{(1)} = \alpha_1 \{(1-\ell_n) - \ell_n (d_1 / d_n)(M_1^{(1)} / M_2^{(1)})\} \tag{6.11a}$$

$$\gamma_{12}^{(1)} = \ell_1 \ell_n (M_1^{(1)} / M_2^{(1)}) / (1-\ell) - \ell_1 \alpha_n (d_n / d_1) \tag{6.11b}$$

$$\mu_{11}^{(1)} = -\ell_1 \ell_n C_2^{(1)} / (1-\ell) - \ell_n \alpha_1 (d_1 / d_n) C_2^{(1)} + (M_3^{(1)} / M_2^{(1)})) \tag{6.12a}$$

$$\mu_{12}^{(1)} = \ell_1 \alpha_n (d_n / d_1) C_1^{(1)} + (1-\ell_1) a_n C_2^{(1)} + \ell_1 \ell_n (M_3^{(1)} / M_2^{(1)}) / (1-\ell) \tag{6.12b}$$

and $W_2^{(1)}$ satisfies the following system:

$$\begin{cases} b_1^{(1)} (w_{22}^{(1)})^2 + b_2^{(1)} w_{22}^{(1)} + b_3^{(1)} = 0 \quad ; \\ w_{22}^{(1)} = -(N_2^{(1)} / N_1^{(1)}) w_{22}^{(1)} - (N_3^{(1)} / N_1^{(1)}); \end{cases} \tag{6.13}$$

$$\begin{cases} v_{21}^{(1)} = A_1^{(1)} + \ell_1 \ell_n (A_1^{(1)} - w_{21}^{(1)}) / (1-\ell) + (d_1 / d_n)\ell_n (1-\ell_1) \\ (A_2^{(1)} - w_{22}^{(1)}) / (1-\ell); \\ v_{22}^{(1)} = w_{22}^{(1)} + \ell_1 \ell_n (w_{22}^{(1)} - A_2^{(1)}) / (1-\ell) + (d_n / d_1)\ell_1 (1-\ell_n) \\ (w_{21}^{(1)} - A_1^{(1)}) / (1-\ell); \end{cases} \tag{6.14}$$

where,

$$b_1^{(1)} = \frac{[((\gamma_{21}^{(1)})^2 - (N_2^{(1)} / N_1^{(1)})^2)c_1^+ - ((\eta_{21}^{(1)})^2 c_1^-)]}{2d_1(1-\ell_1)} + \frac{[((\gamma_{22}^{(1)})^2 c_n^- - (\eta_{22}^{(1)})^2 c_n^+)]}{2d_n(1-\ell_n)} \tag{6.15a}$$

$$b_2^{(1)} = \frac{(\gamma_{21}^{(1)}\mu_{21}^{(1)} - (N_2^{(1)}N_3^{(1)} / (N_1^{(1)})^2)c_1^+ - (\eta_{21}^{(1)})(v_{21}^{(1)} - \delta_1 d_1)c_1^-}{d_1(1-\ell_1)} + \frac{(\gamma_{22}^{(1)}(\mu_{22}^{(1)} - \delta_n d_n) + \delta_n d_n)c_n^- - \eta_{22}^{(1)} v_{22}^{(1)} c_1^-)]}{d_n(1-\ell_n)} \quad (6.15b)$$

$$b_3^{(1)} = \frac{[((\mu_{21}^{(1)})^2 - (N_3^{(1)} / N_1^{(1)})^2 - (I_1^{(1)})^2))c_1^+ - (v_{21}^{(1)} - \delta_1 d_1)^2 c_1^-]}{2d_1(1-\ell_1)} + \frac{\mu_{22}^{(1)}(\mu_{22}^{(1)} - 2\delta_n d_n)^2 c_n^- - (v_{22}^{(1)})^2 - (I_2^{(1)})^2 c_n^+)]}{2d_n(1-\ell_n)} \quad (6.15c)$$

$$\eta_{21}^{(1)} = \alpha_1\left\{(d_1/d_n)\ell_n - (N_2^{(1)}/N_1^{(1)})(1-\ell_n)\right\}; \quad (6.16a)$$

$$\eta_{22}^{(1)} = \ell_1\alpha_n(d_n/d_1)(N_2^{(1)}/N_1^{(1)}) - \ell_1\ell_n/(1-\ell); \quad (6.16b)$$

$$v_{21}^{(1)} = -\ell_1\ell_n C_1^{(1)}/(1-\ell) - \ell_n\alpha_1(d_1/d_n)C_2^{(1)} - (1-\ell_n)\alpha_1(N_3^{(1)}/N_1^{(1)}) - \alpha_1(d_1/d_n)\delta_n d_n; \quad (6.17a)$$

$$v_{22}^{(1)} = \ell_1\alpha_n(d_n/d_1)(C_1^{(1)} + (N_3^{(1)}/N_1^{(1)})) + (1-\ell_1)\alpha_n C_2^{(1)} + \delta_n d_n \ell_1/(1-\ell); \quad (6.17b)$$

$$\gamma_{21}^{(1)} = \ell_1\ell_n(N_2^{(1)}/N_1^{(1)})/(1-\ell) - \ell_n\alpha_1(d_1/d_n); \quad (6.18a)$$

$$\gamma_{22}^{(1)} = \alpha_n\{(1-\ell) - \ell_1(d_n/d_1)(N_2^{(1)}/N_1^{(1)})\}; \quad (6.18b)$$

$$\mu_{21}^{(1)} = (1-\ell_2)\alpha_1 A_1^{(1)} + \ell_n\alpha_1(d_1/d_n)A_2^{(1)} + \ell_n\ell_1(N_3^{(1)}/N_1^{(1)})/(1-\ell); \quad (6.19a)$$

$$\mu_{22}^{(1)} = -\ell_1\alpha_n(d_1/d_n)(A_1^{(1)} + (N_3^{(1)}/N_1^{(1)})) - \ell_1\ell_n A_2^{(1)}/(1-\ell). \quad (6.19b)$$

Notice that in (6.06), $w_{11}^{(1)}$ must be a real root of the quadratic such that $w_{11}^{(1)} \le C_1^{(1)}$.

Similarly, $w_{22}^{(1)}$ in (6.13) must be a real root such that $w_{22}^{(1)} \le A_1^{(1)}$. If either quadratic in (6.06) or (6.13) has no root, then the corridor window does not exist in this case meaning that it is always optimal to reach the extracted schedule form toe their side (i.e. the side with the corridor window).

5.3.7 For Region R_2^u

Equations of boundaries $B_1^{(2)}$ *and* $B_2^{(2)}$:

$$B_1^{(2)} = M_1^{(2)}x_2 + M_2^{(2)}x_3 + M_3^{(2)} = 0; \quad (6.20)$$

$$M_1^{(2)} = c_2^-/(1-\ell_2) - \ell_2\ell_3 c_2^+/(1-\ell_2)(1-\ell) - (d_2/d_3)(\ell_2/\ell_3)\alpha_3 c_3^-; \quad (6.21a)$$

$$M_2^{(2)} = -(d_2/d_3)\ell_3 c_2^+/(1-\ell) - c_2^+/\ell_3 - \alpha_3(1-\ell_3)c_3^-/\ell_3; \quad (6.21b)$$

$$M_3^{(2)} = -(M_1^{(2)}C_1^{(2)} + M_2^{(2)}C_2^{(2)}). \quad (6.21c)$$

and

$$B_2^{(2)} = N_1^{(2)} x_2 + N_2^{(2)} x_3 + N_3^{(2)} = 0; \tag{6.22}$$

$$N_1^{(2)} = -(d_3 / d_2)\ell_2 c_3^+ / (1-\ell) - c_2^+ / \ell_2 - \alpha_2 (1-\ell_3) c_2^- / \ell_2; \tag{6.23a}$$

$$N_2^{(2)} = c_3^- / (1-\ell_3) - \ell_2 \ell_3 c_3^+ / (1-\ell_3)(1-\ell) - (d_3 / d_2)(\ell_3 / \ell_2)\alpha_3 c_2^-; \tag{6.23b}$$

$$N_3^{(2)} = -(N_1^{(2)} A_1^{(2)} + N_2^{(2)} A_2^{(2)}). \tag{6.23c}$$

Corridor windows are:

Let

$$W_1^{(2)} = (w_{11}^{(2)}, w_{12}^{(2)}), V_1^{(2)} = (v_{11}^{(2)}, v_{12}^{(2)})$$

$$W_2^{(2)} = (w_{21}^{(2)}, w_{22}^{(2)}), V_2^{(2)} = (v_{21}^{(2)}, v_{22}^{(2)})$$

In x – space, $W_1^{(2)}$ satisfies the following system:

$$\begin{cases} a_1^{(2)} (w_{11}^{(2)})^2 + a_2^{(2)} w_{11}^{(2)} + a_3^{(2)} = 0 & ; \\ w_{12}^{(2)} = -(M_1^{(2)} / M_2^{(2)}) w_{11}^{(2)} - (M_3^{(2)} / M_2^{(2)}); \end{cases} \tag{6.24}$$

$$\begin{cases} v_{11}^{(2)} = w_{11}^{(2)} + \ell_2 \ell_3 (w_{11}^{(2)} - C_1^{(2)}) / (1-\ell) + (d_2 / d_3)\ell_3 (1-\ell_2) \\ (w_{12}^{(2)} - C_2^{(2)}) / (1-\ell); \\ v_{12}^{(2)} = C_2^{(2)} + \ell_2 \ell_3 (C_2^{(2)} - w_{12}^{(2)}) / (1-\ell) + (d_3 / d_2)\ell_2 (1-\ell_3) \\ (C_1^{(2)} - w_{11}^{(2)}) / (1-\ell); \end{cases} \tag{6.25}$$

where

$$a_1^{(2)} = \frac{((\gamma_{11}^{(2)})^2 - 1)c_2^- - (\eta_{11}^{(2)})^2 c_2^+}{2d_2(1-\ell_2)} + \frac{((\gamma_{12}^{(2)})^2 - (M_1^{(1)} / M_2^{(1)})^2)c_3^+ - (\eta_{12}^{(1)})^2 c_3^-}{2d_3(1-\ell_3)} \tag{6.26a}$$

$$a_2^{(2)} = \frac{(\gamma_{11}^{(2)}(\mu_{11}^{(2)} - \delta_2 d_2) + \delta_2 d_2)c_2^- - \eta_{11}^{(2)} v_{11}^{(2)} c_2^+}{d_2(1-\ell_2)} + \frac{(\gamma_{12}^{(2)} \mu_{12}^{(2)} - M_1^{(2)} M_3^{(2)} / (M_2^{(2)})^2)c_3^+ - \eta_{12}^{(2)} (v_{12}^{(2)} - \delta_3 d_3)c_3^-}{d_3(1-\ell_3)} \tag{6.26b}$$

$$a_3^{(2)} = \frac{\mu_{11}^{(2)}(\mu_{11}^{(2)} - 2\delta_2 d_2)c_2^- - ((v_{11}^{(2)})^2 - (I_2^{(2)})^2)c_2^+}{2d_2(1-\ell_2)} + \frac{((\mu_{12}^{(2)})^2 - (M_3^{(2)} / M_2^{(2)})^2 - (I_2^{(2)})^2)c_3^+ - (v_{12}^{(2)} - \delta_3 d_3)^2 c_3^-}{2d_3(1-\ell_3)} - k_2 \tag{6.26c}$$

$$\eta_{11}^{(2)} = \ell_3 (d_2 / d_3)(M_1^{(2)} / M_2^{(2)})\alpha_2 - \ell_2 / (1-\ell) \tag{6.27a}$$

$$\eta_{12}^{(2)} = \alpha_3\{(d_3 / d_2)\ell_2 - (M_1^{(2)} / M_2^{(2)})(1-\ell_2)\} \tag{6.27b}$$

$$v_{11}^{(2)} = \alpha_2(1-\ell_3)A_1^{(2)} + \alpha_2\ell_3(d_2 / d_3)(A_2^{(2)} + (M_3^{(2)} / M_2^{(2)})) + \delta_2 d_2\ell_3 / (1-\ell) \tag{6.28a}$$

$$v_{12}^{(2)} = -\ell_2\alpha_3(d_3 / d_2)A_1^{(2)} - \ell_2\ell_3A_2^{(2)} / (1-\ell) - (1-\ell_3)\alpha_2(M_3^{(2)} / M_2^{(2)}) - \alpha_3(d_3 / d_2)\delta_2 d_2 \tag{6.28b}$$

$$\gamma_{11}^{(2)} = \alpha_2\{(1-\ell_3) - \ell_3(d_2 / d_3)(M_1^{(2)} / M_2^{(2)})\} \tag{6.29a}$$

$$\gamma_{12}^{(2)} = \ell_2\ell_3(M_1^{(2)} / M_2^{(2)}) / (1-\ell) - \ell_2\alpha_3(d_3 / d_2) \tag{6.29b}$$

$$\mu_{11}^{(2)} = -\ell_2\ell_3C_2^{(2)} / (1-\ell) - \ell_3\alpha_2(d_2 / d_3)C_2^{(2)} + (M_3^{(2)} / M_2^{(2)})) \tag{6.30a}$$

$$\mu_{12}^{(2)} = \ell_2\alpha_3(d_3 / d_2)C_1^{(2)} + (1-\ell_2)a_3^{(2)}C_2^{(2)} + \ell_2\ell_3(M_3^{(2)} / M_2^{(2)}) / (1-\ell) \tag{6.30b}$$

and $w_2^{(2)}$ satisfies the following system:

$$\begin{cases} b_1^{(2)}(w_{22}^{(2)})^2 + b_2^{(2)}w_{22}^{(2)} + b_3^{(2)} = 0 \qquad ; \\ w_{22}^{(2)} = -(N_2^{(2)} / N_1^{(2)})w_{22}^{(2)} - (N_3^{(2)} / N_1^{(2)}); \end{cases} \tag{6.31}$$

$$\begin{cases} v_{21}^{(2)} = A_1^{(2)} + \ell_2\ell_3(A_1^{(2)} - w_{21}^{(2)}) / (1-\ell) + (d_2 / d_3)\ell_3(1-\ell_2) \\ (A_2^{(2)} - w_{22}^{(2)}) / (1-\ell); \\ v_{22}^{(2)} = w_{22}^{(2)} + \ell_2\ell_3(w_{22}^{(2)} - A_2^{(2)}) / (1-\ell) + (d_3 / d_2)\ell_2(1-\ell_3) \\ (w_{21}^{(2)} - A_1^{(2)}) / (1-\ell); \end{cases} \tag{6.32}$$

where,

$$b_1^{(2)} = \frac{[((\gamma_{21}^{(2)})^2 - (N_2^{(2)} / N_1^{(2)})^2)c_2^+ - ((\eta_{21}^{(2)})^2 c_2^-)]}{2d_2(1-\ell_2)} + \frac{[((\gamma_{22}^{(2)})^2 c_3^- - (\eta_{22}^{(2)})^2 c_3^+)]}{2d_3(1-\ell_3)} \tag{6.33a}$$

$$b_2^{(2)} = \frac{(\gamma_{21}^{(2)}\mu_{21}^{(2)} - (N_2^{(2)}N_3^{(2)} / (N_1^{(2)})^2)c_2^+ - (\eta_{21}^{(2)})(v_{21}^{(2)} - \delta_2 d_2)c_2^-}{d_2(1-\ell_2)} + \frac{(\gamma_{22}^{(2)}(\mu_{22}^{(2)} - \delta_3 d_3) + \delta_3 d_3)c_3^- - \eta_{22}^{(2)}v_{22}^{(2)}c_2^-)]}{d_3(1-\ell_3)} \tag{6.33b}$$

$$b_3^{(2)} = \frac{[((\mu_{21}^{(2)})^2 - (N_3^{(2)} / N_1^{(2)})^2 - (I_1^{(2)})^2))c_2^+ - (v_{21}^{(2)} - \delta_2 d_2)^2 c_2^-]}{2d_2(1-\ell_2)} + \frac{\mu_{22}^{(2)}(\mu_{22}^{(2)} - 2\delta_3 d_3)^2 c_3^- - (v_{22}^{(2)})^2 - (I_2^{(2)})^2 c_3^+)]}{2d_3(1-\ell_3)} \tag{6.33c}$$

$$\eta_{21}^{(2)} = \alpha_2\left\{(d_2 / d_3)\ell_3 - (N_2^{(2)} / N_1^{(2)})(1-\ell_3)\right\}; \tag{6.34a}$$

$$\eta_{22}^{(2)} = \ell_2\alpha_3(d_3 / d_2)(N_2^{(2)} / N_1^{(2)}) - \ell_2\ell_3 / (1-\ell); \tag{6.34b}$$

$$v_{21}^{(2)} = -\ell_2\ell_3C_1^{(2)} / (1-\ell) - \ell_3\alpha_2(d_2 / d_3)C_2^{(2)} - (1-\ell_3)\alpha_2(N_3^{(2)} / N_1^{(2)}) - \alpha_2(d_2 / d_3)\delta_3 d_3; \tag{6.35a}$$

$$v_{22}^{(2)} = \ell_2\alpha_3(d_3/d_2)(C_1^{(2)} + (N_3^{(2)}/N_1^{(2)})) + (1-\ell_2)\alpha_3 C_2^{(2)} + \delta_3 d_3 \ell_2/(1-\ell); \tag{6.35b}$$

$$\gamma_{21}^{(2)} = \ell_2\ell_3(N_2^{(2)}/N_1^{(2)})/(1-\ell) - \ell_3\alpha_2(d_2/d_3); \tag{6.36a}$$

$$\gamma_{22}^{(2)} = \alpha_3\{(1-\ell) - \ell_2(d_3/d_2)(N_2^{(2)}/N_1^{(2)})\}; \tag{6.36b}$$

$$\mu_{21}^{(2)} = (1-\ell_3)\alpha_2 A_1^{(2)} + \ell_3\alpha_2(d_2/d_3)A_2^{(2)} + \ell_3\ell_2(N_3^{(2)}/N_1^{(2)})/(1-\ell); \tag{6.37a}$$

$$\mu_{22}^{(2)} = -\ell_2\alpha_3(d_2/d_3)(A_1^{(2)} + (N_3^{(2)}/N_1^{(2)})) - \ell_2\ell_3 A_2^{(2)}/(1-\ell). \tag{6.37b}$$

Notice that in (6.31), $w_{11}^{(2)}$ must be a real root of the quadratic such that $w_{11}^{(2)} \le C_1^{(2)}$.

Similarly, $w_{22}^{(2)}$ in (6.31) must be a real root such that $w_{22}^{(2)} \le A_2^{(2)}$. If either quadratic in (6.22) or (6.31) has no root, then the corridor window does not exist in this case meaning that it is always optimal to reach the extracted schedule form toe their side (i.e. the side with the corridor window).

We continue carrying out the definition of other regions until we get to Region $R_{[n(n-1)/2]}^u$. We define $R_{[n(n-1)/2]}^u$ as follows:

5.3.8 For Region $R_{[n(n-1)/2]}^u$

$$B_1^{(r)} = M_1^{(r)}x_{n-1} + M_2^{(r)}x_n + M_3^{()} = 0; \tag{6.38}$$

$$M_1^{(r)} = c_{n-1}^-/(1-\ell_{n-1}) - \ell_{n-1}\ell_n c_{n-1}^+/(1-\ell_{n-1})(1-\ell) - (d_{n-1}/d_n)(\ell_{n-1}/\ell_n)\alpha_n c_n^- \tag{6.39a}$$

$$M_2^{(r)} = -(d_{n-1}/d_n)\ell_n c_{n-1}^+/(1-\ell) - c_n^+/\ell_n - \alpha_n(1-\ell_n)c_n^-/\ell_n \tag{6.39b}$$

$$M_3^{(r)} = -(M_1^{(r)}C_1^{(r)} + M_2^{(r)}C_2^{(r)}) \tag{6.39c}$$

and

$$B_2^{(r)} = N_1^{(r)}x_{n-1} + N_2^{(r)}x_n + N_3^{(r)} = 0; \tag{6.40}$$

$$N_1^{(r)} = -(d_n/d_{n-1})\ell_{n-1}c_n^+/(1-\ell) - c_{n-1}^+/\ell_{n-1} - \alpha_{n-1}(1-\ell_n)c_{n-1}^-/\ell_{n-1}; \tag{6.41a}$$

$$N_2^{(r)} = c_n^-(1-\ell_n) - \ell_{n-1}\ell_n c_n^+/(1-\ell_n)(1-\ell) - (d_n/d_{n-1})(\ell_n/\ell_{n-1})\alpha_{n-1}c_{n-1}^-/\ell_{n-1}; \tag{6.41b}$$

$$N_3^{(r)} = -(N_1^{(r)}A_1^{(r)} + N_2^{(r)}A_2^{(r)}) \tag{6.41c}$$

Corridor windows are:

Let

$$W_1^{(r)} = (w_{11}^{(r)}, w_{12}^{(r)}), V_1^{(r)} = (v_{11}^{(r)}, v_{12}^{(r)})$$

$$W_2^{(r)} = (w_{21}^{(r)}, w_{22}^{(r)}), V_2^{(r)} = (v_{21}^{(r)}, v_{22}^{(r)})$$

in *x-space*. $W_1^{(r)}$ satisfies the following system:

$$\begin{cases} a_1^{(r)}(w_{11}^{(r)})^2 + a_2^{(r)}w_{11}^{(r)} + a_3^{(r)} = 0 & ; \\ w_{12}^{(r)} = -(M_1^{(r)} / M_2^{(r)})w_{11}^{(r)} - (M_3^{(r)} / M_2^{(r)}); \end{cases} \tag{6.42}$$

$$\begin{cases} v_{11}^{(r)} = w_{11}^{(r)} + \ell_{n-1}\ell_n(w_{11}^{(r)} - C_1^{(r)}) / (1-\ell) + (d_{n-1} / d_n)\ell_n(1-\ell_{n-1}) \\ (w_{12}^{(r)} - C_2^{(r)}) / (1-\ell); \\ v_{12}^{(r)} = C_2^{(r)} + \ell_{n-1}\ell_n(C_2^{(r)} - w_{12}^{(r)}) / (1-\ell) + (d_n / d_{n-1})\ell_{n-1}(1-\ell_n) \\ (C_1^{(r)} - w_{11}^{(r)}) / (1-\ell); \end{cases} \tag{6.43}$$

where

$$a_1^{(r)} = \frac{((\gamma_{11}^{(r)})^2 - 1)c_{n-1}^- - (\eta_{11}^{(r)})^2 c_{n-1}^+}{2d_{n-1}(1-\ell_{n-1})} + \frac{((\gamma_{12}^{(r)})^2 - (M_1^{(r)} / M_2^{(r)})^2)c_n^+ - (\eta_{12}^{(r)})^2 c_n^-}{2d_n(1-\ell_n)} \tag{6.44a}$$

$$a_2^{(r)} = \frac{(\gamma_{11}^{(r)}(\mu_{11}^{(r)} - \delta_{n-1}d_{n-1}) + \delta_{n-1}d_{n-1})c_{n-1}^- - \eta_{11}^{(r)}v_{11}^{(r)}c_{n-1}^+}{d_{n-1}(1-\ell_{n-1})} + \frac{(\gamma_{12}^{(r)}\mu_{12}^{(r)} - M_1^{(r)}M_3^{(r)} / (M_2^{(r)})^2)c_n^+ - \eta_{12}^{(r)}(v_{12}^{(r)} - \delta_n d_n)c_n^-}{d_n(1-\ell_n)} \tag{6.44b}$$

$$a_3^{(r)} = \frac{\mu_{11}^{(r)}(\mu_{11}^{(r)} - 2\delta_{n-1}d_{n-1})c_{n-1}^- - ((v_{11}^{(r)})^2 - (I_2^{(r)})^2)c_{n-1}^+}{2d_{n-1}(1-\ell_{n-1})} + \frac{((\mu_{12}^{(r)})^2 - (M_3^{(r)} / M_2^{(r)})^2 - (I_2^{(r)})^2)c_n^+ - (v_{12}^{(r)} - \delta_n d_n)^2 c_n^-}{2d_n(1-\ell_n)} - k_n \tag{6.44c}$$

$$\eta_{11}^{(r)} = \ell_n(d_{n-1} / d_n)(M_1^{(r)} / M_2^{(r)})\alpha_{n-1} - \ell_{n-1} / (1-\ell) \tag{6.45a}$$

$$\eta_{12}^{(r)} = \alpha_n\{(d_n / d_{n-1})\ell_{n-1} - (M_1^{(r)} / M_2^{(r)})(1-\ell_{n-1})\} \tag{6.45b}$$

$$v_{11}^{(r)} = \alpha_{n-1}(1-\ell_n)A_1^{(r)} + \alpha_{n-1}\ell_n(d_{n-1} / d_n)(A_2^{(r)} + (M_3^{(r)} / M_2^{(r)})) + \delta_{n-1}d_{n-1}\ell_n / (1-\ell) \tag{6.46a}$$

$$v_{12}^{(r)} = -\ell_{n-1}\alpha_n(d_n / d_{n-1})A_1^{(r)} - \ell_{n-1}\ell_n A_2^{(r)} / (1-\ell) - (1-\ell_n)\alpha_{n-1}(M_3^{(r)} / M_2^{(r)}) - \alpha_n(d_n / d_{n-1})\delta_{n-1}d_{n-1} \tag{6.46b}$$

$$\gamma_{11}^{(r)} = \alpha_{n-1}\{(1-\ell_n) - \ell_n(d_{n-1} / d_n)(M_1^{(r)} / M_2^{(r)})\} \tag{6.47a}$$

$$\gamma_{12}^{(r)} = \ell_{n-1}\ell_n (M_1^{(r)} / M_2^{(r)}) / (1-\ell) - \ell_{n-1}\alpha_n (d_n / d_{n-1}) \tag{6.47b}$$

$$\mu_{11}^{(r)} = -\ell_{n-1}\ell_n C_2^{(r)} / (1-\ell) - \ell_n \alpha_{n-1} (d_{n-1} / d_n) C_2^{(r)} + (M_3^{(r)} / M_2^{(r)})) \tag{6.48a}$$

$$\mu_{12}^{(r)} = \ell_{n-1}\alpha_n (d_n / d_{n-1}) C_1^{(r)} + (1-\ell_{n-1}) a_n C_2^{(r)} + \ell_{n-1}\ell_n (M_3^{(r)} / M_2^{(r)}) / (1-\ell) \tag{6.48b}$$

and $W_2^{(r)}$ satisfies the following system:

$$\begin{cases} b_1^{(r)} (w_{22}^{(r)})^2 + b_2^{(r)} w_{22}^{(r)} + b_3^{(r)} = 0 \qquad ; \\ w_{22}^{(r)} = -(N_2^{(r)} / N_1^{(r)}) w_{22}^{(r)} - (N_3^{(r)} / N_1^{(r)}); \end{cases} \tag{6.49}$$

$$\begin{cases} v_{21}^{(r)} = A_1^{(r)} + \ell_{n-1}\ell_n (A_1^{(r)} - w_{21}^{(r)}) / (1-\ell) + (d_{n-1} / d_n) \ell_n (1-\ell_{n-1}) \\ (A_2^{(r)} - w_{22}^{(r)}) / (1-\ell); \\ v_{22}^{(r)} = w_{22}^{(r)} + \ell_{n-1}\ell_n (w_{22}^{(r)} - A_2^{(r)}) / (1-\ell) + (d_n / d_{n-1}) \ell_{n-1} (1-\ell_n) \\ (w_{21}^{(r)} - A_1^{(r)}) / (1-\ell); \end{cases} \tag{6.50}$$

where,

$$b_1^{(r)} = \frac{[((\gamma_{21}^{(r)})^2 - (N_2^{(r)} / N_1^{(r)})^2) c_{n-1}^+ - ((\eta_{21}^{(r)})^2 c_{n-1}^-)]}{2d_{n-1}(1-\ell_{n-1})} + \frac{[((\gamma_{22}^{(r)})^2 c_n^- - (\eta_{22}^{(r)})^2 c_n^+)]}{2d_n(1-\ell_n)} \tag{6.51a}$$

$$b_2^{(r)} = \frac{(\gamma_{21}^{(r)}\mu_{21}^{(r)} - (N_2^{(r)} N_3^{(r)} / (N_1^{(r)})^2) c_{n-1}^+ - (\eta_{21}^{(r)})(v_{21}^{(r)} - \delta_{n-1} d_{n-1}) c_{n-1}^-}{d_{n-1}(1-\ell_{n-1})} + \frac{(\gamma_{22}^{(r)}(\mu_{22}^{(r)} - \delta_n d_n) + \delta_n d_n) c_n^- - \eta_{22}^{(r)} v_{22}^{(r)} c_n^+)]}{d_n(1-\ell_n)} \tag{6.51b}$$

$$b_3^{(r)} = \frac{[((\mu_{21}^{(r)})^2 - (N_3^{(r)} / N_1^{(r)})^2 - (I_1^{(r)})^2)) c_{n-1}^+ - (v_{21}^{(r)} - \delta_{n-1} d_{n-1})^2 c_{n-1}^-]}{2d_{n-1}(1-\ell_{n-1})} + \frac{\mu_{22}^{(r)}(\mu_{22}^{(r)} - 2\delta_n d_n)^2 c_n^- - (v_{22}^{(r)})^2 - (I_2^{(r)})^2 c_n^+)]}{2d_n(1-\ell_n)} \tag{6.51c}$$

$$\eta_{21}^{(r)} = \alpha_{n-1} \left\{ (d_{n-1} / d_n) \ell_n - (N_2^{(r)} / N_1^{(r)})(1-\ell_n) \right\}; \tag{6.52a}$$

$$\eta_{22}^{(r)} = \ell_{n-1}\alpha_n (d_n / d_{n-1})(N_2^{(r)} / N_1^{(r)}) - \ell_{n-1}\ell_n / (1-\ell); \tag{6.52b}$$

$$v_{21}^{(r)} = -\ell_{n-1}\ell_n C_1^{(r)} / (1-\ell) - \ell_n \alpha_{n-1} (d_{n-1} / d_n) C_2^{(r)} - (1-\ell_n) \alpha_{n-1} (N_3^{(r)} / N_1^{(r)}) - \alpha_{n-1} (d_{n-1} / d_n) \delta_n d_n; \tag{6.53a}$$

$$v_{22}^{(r)} = \ell_{n-1}\alpha_n (d_n / d_{n-1})(C_1^{(r)} + (N_3^{(r)} / N_1^{(r)})) + (1-\ell_{n-1}) \alpha_n C_2^{(r)} + \delta_n d_n \ell_{n-1} / (1-\ell); \tag{6.53b}$$

$$\gamma_{21}^{(r)} = \ell_{n-1}\ell_n (N_2^{(r)} / N_1^{(r)}) / (1-\ell) - \ell_n \alpha_{n-1} (d_{n-1} / d_n); \tag{6.54a}$$

$$\gamma_{22}^{(r)} = \alpha_n \{ (1-\ell) - \ell_{n-1} (d_n / d_{n-1})(N_2^{(r)} / N_1^{(r)}) \}; \tag{6.54b}$$

$$\mu_{21}^{(r)} = (1-\ell_n) \alpha_{n-1} A_1^{(r)} + \ell_n \alpha_{n-1} (d_{n-1} / d_n) A_2^{(r)} + \ell_n \ell_{n-1} (N_3^{(r)} / N_1^{(r)}) / (1-\ell); \tag{6.55a}$$

$$\mu_{22}^{(r)} = -\ell_{n-1}\alpha_n (d_{n-1} / d_n)(A_1^{(r)} + (N_3^{(r)} / N_1^{(r)})) - \ell_{n-1}\ell_n A_2^{(r)} / (1-\ell). \qquad (6.55b)$$

Notice that in (6.42), $w_{11}^{(r)}$ must be a real root of the quadratic such that $w_{11}^{(r)} \le C_1^{(r)}$.

Similarly, $w_{22}^{(r)}$ in (6.49) must be a real root such that $w_{22}^{(r)} \le A_1^{(r)}$. If either quadratic in (6.42) or (6.49) has no root, then the corridor window does not exist in this case meaning that it is always optimal to reach the extracted schedule form toe their side (i.e. the side with the corridor window).

5.3.9 Optimal Trajectory in Region $R^u(R_1^u, R_2^u, ..., R_{[n(n-1)/2]}^u)$

Based on Facts 1 and 2, the Optimal Trajectory emanating from a point in Region $R^u(R_1^u, R_2^u, ..., R_{[n(n-1)/2]}^u)$ and leading to the extracted schedule in finite time can be obtained as follows: Given an initial surplus point in each of Region Region $R^u(R_1^u, R_2^u, ..., R_{[n(n-1)/2]}^u)$; the first setup is chosen and the cost of the trajectory leading to the extracted schedule with two setup changes only is calculated. At this point; in each case, there is a $D_i - 2 - step$ trajectory, where i is the initial setup for product Type i. Next, attempt was made to lower the cost of the current trajectory by introducing a setup change to the other product Type so that the extracted schedule is reached at the opposite side.

Therefore, if the cost can be reduced, then the extracted schedule is read at the opposite side.

Therefore, if the cost can be reduced, that the obtained new trajectory is a $D_i - 3 - step$ trajectory. By keeping the trying to reduce the cost of the current trajectory by introducing each time, a setup change before the extracted schedule is reached until the cost cannot be lowered anymore. At this point, there is an optimal $D_i - n - step$ trajectory emanating in each Region $R^u(R_1^u, R_2^u, ..., R_{[n(n-1)/2]}^u)$ and reaching the extracted schedule in finite time. Similarly, the optimal $D_i - n - step$ [where $(j \ne i \ne l \ne .... \ne z)$] trajectories are obtained starting with a setup to product Type j, $l, \ldots, z$ first in each case respectively. Thus, the optimal trajectory would be the one with the lowest cost.

Bai and Elhafsi (1996) and Onanaye (2006) with the numerical solutions of various examples, suggest that the above procedure be further simplified based on the following conjecture.

CONJECTURES

The initial setup of the optimal trajectory is given by $i \neq \arg\min_{i=1,2}\left\{C_i^2(x)\right\}$; is the cost of the $D_i - 2 - step$ trajectory staring with a setup change to product Type *i* and reaching the extracted schedule with two setup changes only. Similar conjecture for product type *j* and *l* with initial setup of the optimal trajectory given by

$i \neq \arg\min_{i=1,2}\left\{C_i^2(x)\right\}$;

$j \neq \arg\min_{j=1,2}\left\{C_j^2(x)\right\}$;

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$z \neq \arg\min_{z=1,2}\left\{C_z^2(x)\right\}$; respectively.

Based on the above Conjectures, the optimal trajectory originating from $X = (a,b)$ in Region $R^u(R_1^u, R_2^u, ..., R_{[n(n-1)/2]}^u)$ is obtained using the following procedures:

PROCEDURE

STEP 0: $X_1(0) := a^{(1)}$, $X_n(0) := b^{(1)}; k := 1$;

$IF\ ((C_1^{(1)})^2 X^{(1)}(0)) \geq ((C_2^{(1)})^2 X^{(1)}(0))\ THEN$

$X_1(k) := X_1(0) - \delta_n d_1$;

$X_n(k) := X_n(0) - \delta_n d_n$;

GOTO STEP 1;

ELSE

$X_1(k) := X_1(0) - \delta_1 d_1$;

$X_n(k) := X_n(0) - \delta_1 d_n$;

ENDIF;

STEP 1: $k := k+1$

$X_1(k) := (d_1\ell_n(M_3^{(1)} + X_n(k-1)M_2^{(1)} + d_2(1-\ell_n)X_1(k-1)M_2^{(1)}) / (d_n(1-\ell_n)M_2^{(1)} - d_1\ell_n M_1^{(1)})$

$X_n(k) := -(M_1^{(1)} / M_2^{(1)})X_1(k) - (M_3^{(1)} / M_2^{(1)})$;

IF $X_1(k) \geq W_{11}^{(1)}$ THEN

$X_1(k) := \gamma_{11}^{(1)} X_1(k-1) + \mu_{11}^{(1)}$

$X_n(k) := \gamma_{12}^{(1)} X_n(k-1) + \mu_{12}^{(1)}$

$k := k+1$

$X(k) := I^{(1)}$

GOTO STEP 3;

ELSE

GOTO STEP 2:

END IF;

STEP 2: $k := k+1$

$X_n(k) := (d_n \ell_1 (N_3^{(1)} + X_1(k-1) N_1^{(1)} + \quad d_1(1-\ell_1) X_1(k-1) N_1^{(1)}) / (d_1(1-\ell_1) N_1^{(1)} - d_n \ell_1 N_2^{(1)})$

$X_n(k) := -(N_2^{(1)} / N_1^{(1)}) X_n(k) - (N_3^{(1)} / N_1^{(1)});$

IF $X_n(k) \geq w_{22}^{(1)}$ THEN

$X_1(k) := \gamma_{21}^{(1)} X_1(k-1) + \mu_{21}^{(1)}$

$X_n(k) := \gamma_{22}^{(1)} X_n(k-1) + \mu_{22}^{(1)}$

$k := k+1$

$X_1(k) := X_1(k-1) - \delta_n d_1;$

$X_n(k) := X_n(k-1) - \delta_n d_n;$

$k := k+1$

$X(k) := I^{(1)}$

GOTO STEP 3;

ELSE

GOTO STEP 1:

END IF;

STEP 3: OUTPUT: OPTIMAL TRAJECOTRY =

$\left\{X^{(1)}(0), X^{(1)}(1), \ldots., X^{(1)}(k-1), X^{(1)}(k)\right\};$

Module 2: $\text{Re}\,gion\ R_2^u$

STEP 0: $X_2(0) := a^{(2)}, X_3(0) := b^{(2)}; k := 1;$

$IF\ (C_1^{(2)})^2 X^{(2)}(0)) \geq (C_2^{(2)})^2 (X^{(2)}(0))$ THEN

$X_2(k) := X_2(0) - \delta_3 d_2;$

$X_3(k) := X_3(0) - \delta_3 d_3;$

GOTO STEP 1;

ELSE

$X_2(k) := X_2(0) - \delta_2 d_2;$

$X_3(k) := X_3(0) - \delta_2 d_3;$

ENDIF;

STEP 1: $k := k+1$

$X_2(k) := (d_2 \ell_3 M_3^{(2)} + X_3(k-1) M_2^{(2)} +$

$d_3(1-\ell_3) X_2(k-1) M_2^{(2)}) / (d_3(1-\ell_3) M_2^{(2)} - d_2 \ell_3 M_2^{(2)});$

$X_3(k) := -(M_1^{(2)} / M_2^{(2)}) X_2(k) - (M_3^{(2)} / M_2^{(2)});$

IF $X_2(k) \geq w_{11}^{(2)}$ THEN

$X_2(k) := \gamma_{11}^{(2)} X_2(k-1) + \mu_{11}^{(2)};$

$X_3(k) := \gamma_{12}^{(2)} X_3(k-1) + \mu_{12}^{(2)};$

$k := k+1$

$X(k) := I^{(2)}$

GOTO STEP 3;

ELSE

GOTO STEP 2:

END IF;

STEP 2: $k := k+1$

$X_3(k) := (d_3 \ell_2 (N_3^{(2)} + X_2(k-1) N_1^{(2)} +$

$d_2(1-\ell_2) X_2(k-1) N_1^{(2)}) / (d_2(1-\ell_2) N_1^{(2)} - d_3 \ell_2 N_2^{(2)});$

$X_2(k) := -(N_2^{(2)} / N_1^{(2)}) X_3(k) - (N_3^{(2)} / N_1^{(2)});$

IF $X_3(k) \geq w_{22}^{(2)}$ THEN

$X_2(k) := \gamma_{21}^{(2)} X_2(k-1) + \mu_{21}^{(2)};$

$X_3(k) := \gamma_{22}^{(2)} X_3(k-1) + \mu_{22}^{(2)};$

$k := k+1$

$X_2(k) := X_2(k-1) - \delta_3 d_2;$

$X_3(k) := X_3(k-1) - \delta_3 d_3;$

$k := k+1$

$X(k) := I^{(2)}$

GOTO STEP 3;

ELSE

GOTO STEP 1:

END IF;

STEP 3: OUTPUT: OPTIMAL TRAJECOTRY =

$\left\{X^{(2)}(0), X^{(2)}(1), \ldots., X^{(2)}(k-1), X^{(2)}(k)\right\}$;

We shall likewise compute Modules for Regions $R_3^u, R_4^u, R_5^u, \ldots, R_{[(n(n-1)/2)-1]}^u$, then lastly we compute for Module *r for* Region $R_{[n(n-1)/2]}^u$ as follows:

Module *r for* Region $R_{[n(n-1)/2]}^u$

STEP 0: $X_{n-1}(0) := a^{(r)}; X_n(0) := b^{(r)}; k := 1;$

$IF\ ((C_1^{(r)})^2 X^{(r)}(0)) \geq ((C_2^{(r)})^2 X^{(r)}(0))\ THEN$

$X_{n-1}(k) := X_{n-1}(0) - \delta_n d_{n-1};$

$X_n(k) := X_n(0) - \delta_n d_n;$

GOTO STEP 1;

ELSE

$X_{n-1}(k) := X_{n-1}(0) - \delta_{n-1} d_{n-1};$

$X_n(k) := X_n(0) - \delta_{n-1} d_n;$

ENDIF;

STEP 1: $k := k+1$

$X_{n-1}(k) := (d_{n-1}\ell_n(M_3^{(r)} + X_n(k-1)M_2^{(r)} +$

$d_n(1-\ell_n)X_{n-1}(k-1)M_2^{(r)}) / (d_n(1-\ell_n)M_2^{(r)} - d_{n-1}\ell_n M_1^{(r)})$

$X_n(k) := -(M_1^{(r)} / M_2^{(r)})X_{n-1}(k) - (M_3^{(r)} / M_2^{(r)});$

IF $X_{n-1}(k) \geq w_{11}^{(r)}$ THEN

$X_{n-1}(k) := \gamma_{11}^{(r)} X_{n-1}(k-1) + \mu_{11}^{(r)};$

$X_n(k) := \gamma_{12}^{(r)} X_n(k-1) + \mu_{12}^{(r)}$

$k := k+1$

$X(k) := I^{(r)}$

GOTO STEP 3;

ELSE

GOTO STEP 2:

END IF;

STEP 2: $k := k+1$

$X_n(k) := (d_n\ell_{n-1}(N_3^{(r)} + X_{n-1}(k-1)N_1^{(r)} +$

$d_{n-1}(1-\ell_{n-1})X_{n-1}(k-1)N_1^{(r)}) / (d_{n-1}(1-\ell_{n-1})N_1^{(r)} - d_n\ell_{n-1}N_2^{(r)})$

$X_n(k) := -(N_2^{(r)} / N_1^{(r)})X_n(k) - (N_3^{(r)} / N_1^{(r)});$

IF $X_n(k) \geq w_{22}^{(r)}$ THEN

$X_{n-1}(k) := \gamma_{21}^{(r)} X_{n-1}(k-1) + \mu_{21}^{(r)};$

$X_n(k) := \gamma_{22}^{(1)} X_n(k-1) + \mu_{22}^{(1)};$

$k := k+1$

$X_{n-1}(k) := X_{n-1}(k-1) - \delta_n d_{n-1};$

$X_n(k) := X_n(k-1) - \delta_n d_n;$

$k := k+1$

$X(k) := I^{(r)}$

GOTO STEP 3;

ELSE

GOTO STEP 1:

END IF;

STEP 3: OUTPUT: OPTIMAL TRAJECOTRY =

$\left\{ X^{(r)}(0), X^{(r)}(1), \ldots., X^{(r)}(k-1), X^{(r)}(k) \right\};$

CHAPTER SIX

SOLVED PROBLEM

6.0 Numerical Example

Consider the following system

Product Type 1: $c_1^+ =$ ₦5/Unit/day, $c_1^- =$ ₦25/Unit/day, U_1=10/day, d_1=3.125/day, δ_1=1day and $k_1 =$ ₦250.00

Product Type 2: $c_2^+ =$ ₦5/Unit/day, $c_2^- =$ ₦25/Unit/day, U_2=12/day, d_2=3.6/day, δ_2=1day and $k_2 =$ ₦300.00

Product Type 3: $c_3^+ =$ ₦5/Unit/day, $c_3^- =$ ₦25/Unit/day, U_3=14/day, d_3=4.075/day, δ_3=1day and $k_3 =$ ₦350.00

Product Type 4: $c_4^+ =$ ₦5/Unit/day, $c_4^- =$ ₦25/Unit/day, U_4=16/day, d_4=4.613/day, δ_4=1day and $k_4 =$ N400.00

Product Type 5: $c_5^+ =$ ₦5/Unit/day, $c_5^- =$ ₦25/Unit/day, U_5=18/day, d_5=5.222/day, δ_5=1day and $k_5 =$ N450.00

6.1 Numerical Solution Explained

The optimal production and setup planning for this system can be described as follows (after rounding up):

Setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches 17 Units; setup the machine and produce product type1 at the rate of 10 Units/day until its surplus level reaches 13 Units; setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches 18 Units; setup the machine and produce product type1 at the rate of 10 Units/day until its surplus level reaches 20 Units.

Switch the control action to the cyclic schedule given as follows: setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches 0; continue producing product type 2 at the demand rate of 5 Units/day until the surplus level of product Type 1 drops to 8 Units; at this moment, increase the production rate of product type 2 to 12 Units/day until its surplus level reaches 19 Units. At this point, setup the machine and produce product type 3 at the rate of 14 Units/day until its surplus level reaches 20 Units; setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches 17 Units; setup the machine and produce product type 3 at the rate of 14 Units/day until its surplus level reaches 21 Units; setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches 18 Units.

Switch to the control action to the cyclic schedule given as follows: setup the machine and produce product type 3 at the rate of 14 Units/day until its surplus level reaches 0; continue producing product type 3 at the demand rate of 4.5 Units/day until the surplus level of product type 2 drops to 11 Units; at this moment, increase the production rate of product type 3 to 14 Units/day until its surplus level reaches 21 Units. Again, we now, setup the machine and produce product type 1 at the rate of 10 Units/day until its surplus level reaches 13 Units; setup the machine and produce product type 3 at the rate of 14 Units/day until its surplus level reaches 20 Units; setup the machine and produce product type 1 at the rate of 10 Units/day until its surplus level reaches 20 Units, setup the machine and produce product type 3 at the rate of 14 Units/day until its surplus level reaches 21 Units.

Switch to the control action to the cyclic schedule given as follows: setup the machine and produce product type 1 at the rate of 10 Units/day until its surplus level reaches 0; continue producing product type 1 at the demand rate of 4 Units/day until the surplus level of product type 3 drops to 14 Units; at this moment, increase the production rate of product type 1 to 10 Units/day until its surplus level reaches 20 Units. Again, we now, setup the machine and produce product type 1 at the of 10 Units/day until its surplus level reaches 13 Units; setup the machine and produce product type 4 at the rate of 16 Units/day until its surplus level reaches 23 Units; setup the machine and produce product type 1 at the rate of 10 Units/day until its surplus level reaches 20 Units, setup the machine and produce product type 4 at the rate of 16 Units/day until its surplus level reaches 32 Units.

Switch to the control action to the cyclic schedule given as follows: setup the machine and produce product type 1 at the rate of 10 Units/day until its surplus level reaches 0; continue producing product type 1 at the demand rate of 3.5 Units/day until the surplus level of product type 4 drops to 21 Units; at this moment, increase the production rate of product type 1 to 10 Units/day until its surplus level reaches 20 Units. Setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches 17 Units; setup the machine and produce product type 4 at the rate of 16 Units/day until its surplus level reaches 23 Units; setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches

18 Units; setup the machine and produce product type4 at the rate of 16 Units/day until its surplus level reaches 24 Units.

Switch the control action to the cyclic schedule given as follows: setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches 0; continue producing product type 2 at the demand rate of 3 Units/day until the surplus level of product Type 4 drops to 13 Units; at this moment, increase the production rate of product type 2 to 12 Units/day until its surplus level reaches 19 Units. Again, we now, setup the machine and produce product type 3 at the of 14 Units/day until its surplus level reaches 20 Units; setup the machine and produce product type 4 at the rate of 16 Units/day until its surplus level reaches 23 Units; setup the machine and produce product type 3 at the rate of 14 Units/day until its surplus level reaches 21 Units, setup the machine and produce product type 4 at the rate of 16 Units/day until its surplus level reaches 24 Units.

Switch to the control action to the cyclic schedule given as follows: setup the machine and produce product type 3 at the rate of 14 Units/day until its surplus level reaches 0; continue producing product type 3 at the demand rate of 2.5 Units/day until the surplus level of product type 4 drops to 15 Units; at this moment, increase the production rate of product type 3 to 14 Units/day until its surplus level reaches 21 Units. Again, we now, setup the machine and produce product type 1 at the of 10 Units/day until its surplus level reaches 15 Units; setup the machine and produce product type 5 at the rate of 18 Units/day until its surplus level reaches 26 Units; setup the machine and produce product type 1 at the rate of 10 Units/day until its surplus level reaches 23 Units, setup the machine and produce product type 5 at the rate of 18 Units/day until its surplus level reaches 36 Units.

Switch to the control action to the cyclic schedule given as follows: setup the machine and produce product type 1 at the rate of 10 Units/day until its surplus level reaches 0; continue producing product type 1 at the demand rate of 3.5 Units/day until the surplus level of product type 5 drops to 24 Units; at this moment, increase the production rate of product type 1 to 10 Units/day until its surplus level reaches 23 Units. Setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches 20 Units; setup the machine and produce product type 5 at the rate of 18 Units/day until its surplus level reaches 26 Units; setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches

21 Units; setup the machine and produce product type 5 at the rate of 18 Units/day until its surplus level reaches 27 Units.

Switch the control action to the cyclic schedule given as follows: setup the machine and produce product type 2 at the rate of 12 Units/day until its surplus level reaches 0; continue producing product type 2 at the demand rate of 3 Units/day until the surplus level of product Type 5 drops to 15 Units; at this moment, increase the production rate of product type 2 to 12 Units/day until its surplus level reaches 22 Units. Again, we now, setup the machine and produce product type 3 at the of 14 Units/day until its surplus level reaches 23 Units; setup the machine and produce product type 5 at the rate of 18 Units/day until its surplus level reaches 26 Units; setup the machine and produce product type 3 at the rate of 14 Units/day until its surplus level reaches 22 Units, setup the machine and produce product type 5 at the rate of 18 Units/day until its surplus level reaches 27 Units.

Switch to the control action to the cyclic schedule given as follows: setup the machine and produce product type 3 at the rate of 14 Units/day until its surplus level reaches 0; continue producing product type 3 at the demand rate of 2.5 Units/day until the surplus level of product type 5 drops to 17 Units; at this moment, increase the production rate of product type 3 to 14 Units/day until its surplus level reaches 24 Units. Again, we now, setup the machine and produce product type 4 at the of 16 Units/day until its surplus level reaches 26 Units; setup the machine and produce product type 5 at the rate of 18 Units/day until its surplus level reaches 26 Units; setup the machine and produce product type 4 at the rate of 16 Units/day until its surplus level reaches 25 Units, setup the machine and produce product type 5 at the rate of 18 Units/day until its surplus level reaches 27 Units.

Switch to the control action to the cyclic schedule given as follows: setup the machine and produce product type 4 at the rate of 16 Units/day until its surplus level reaches 0; continue producing product type 4 at the demand rate of 2.8 Units/day until the surplus level of product type 5 drops to 19 Units; at this moment, increase the production rate of product type 4 to 14 Units/day until its surplus level reaches 27 Units.

Now, we have come to the end of one compLete cyclic schedule. The machine is setup again to start with product type 2 and the processes continue on and on.

CHAPTER SEVEN
SUMMARY, RECOMMENDATION AND CONCLUSION

7.0 Summary

In this work, a study was carried out on a multiple-product manufacturing system incurring setup times as well as setup costs when production is switched from one product to the other. The production rates are controllable.

A provision was made for a feedback control formulation of the problem. Two different periods: a steady state period (for which the problem reduces to the multiple-product Economic Lot Scheduling Problem), and a transient period; were highlighted and distinguished between the two. The steady state optimal solution consists of a cyclic schedule, where the *n* products are produced alternatively. It was shown that in the steady state, within the production run of a product, it might be optimal to reduce the production rate from the maximum of the demand rate so as to keep its inventory at the zero level for a certain amount of time, and therefore, delay as much as possible the setup cost of the next setup change. This is particularly true in the case of high setup costs. This will achieve the aim of not wanting to switch setups frequently.

On the other hand, the optimal transient solution consists of a trajectory in *x-space* leading to the cyclic schedule in finite time and with minimum cost.

This was obtained by partitioning the *x-space* into $n(n-1)$ mutually exclusive major regions. For initial surplus levels (inventory/backlog) in the region $\left(R_1^o, R_2^o, R_3^o, \ldots, R_{[n(n-1)/2]}^o\right)$ the optimal solutions are obtained by inspection. For initial surplus levels in other regions $\left(R_1^u, R_2^u, R_3^u, \ldots, R_{[n(n-1)/2]}^o\right)$ a procedure was provided for each region, to determine the optimal state-space trajectory.

7.1 Recommendation

From the result obtained in this work, it is obvious that the application of applied Mathematics (Optimization Techniques) in solving production related problems cannot be over-emphasized. We thereby put forward the following recommendations:

1. That Optimization technique-Optimal production control, as an essential tool for today's production or operations managers (Engineers) in manufacturing society. And as result they should be involved.
2. That companies which produce multiple products yet from the same machine, should embrace the Optimal production control of a dynamic n-product manufacturing system with Setup Costs and Setup Time;

this eliminate un-necessary wastage of resources and reduce cost of production and to meet the customers’ demands and satisfaction.

7.2 Conclusion

In conclusion, the goal of the Production and Setup Scheduling Problem (PSSP) is to minimize the total backlog, inventory and setup cost incurred over a finite horizon. The optimal solution provides the optimal production rate and setup switching epochs as a function of the state of the system (backlog and inventory levels).

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APPENDIX A1

Proof of Theorem 1

This proof is based on the Hamilton-Jacobi-Bellman (HJB) equation. Throughout the proof, it is assumed that the optimal cost functional is differentiable in *x* and *t;* in fact, the optimal state trajectory is continuous piece-wise linear.

Hence, the optimal cost will not depend explicitly on *t* and will sum of quadratics in *x* (since the cost rate is linear in *x*) and therefore differentiable in *x*.

Let $J_T^*(x,t) = \min_{\sigma,u\in\psi(\phi,\Omega)} \int_t^{T_f} g(x(s),\sigma(s))ds$ (A1.0)

Where T_f is finite

We proof the transient case first. Then using a similar argument, the steady state case is proved.

Transient case: In this case, Let $T_f = t_f$ in the expression of $J_{T_f}^*$ to obtain the optimal transient cost component.

The HJB equation (see Gershwin (1993) for a formal derivation) is given by:

$$J^* = \min_{\sigma,u\in\psi(\phi,\Omega)} \left\{ g(x,\sigma) + \frac{\partial J_{T_f}^*(x,t)}{\partial x_1}(u_1 - d_1) + \ldots + \frac{\partial J_{T_f}^*(x,t)}{\partial x_n}(u_n - d_n) \right\} \quad \text{(A1.1)}$$

It is clear that, when the machine is undergoing a setup change to a product Type, there is no decision to make end and $(u_1^*, u_2^*, u_3^*, \ldots, u_n^*)$ is forced to be equal to $(0,0,0,\ldots,0)$.

Now, assume that the optimal setup state of the machine is known and $\sigma = (1,0,0,0,0,\ldots,0)$ be this Type 1, then the HJB equation can then be written as follows:

$$\frac{\partial J_{T_f}^*(x,t)}{\partial t} = \min_{\sigma,u\in\Omega}(1,0,0,\ldots,0) \left\{ g(x,\sigma) + \frac{\partial J_{T_f}^*(x,t)}{\partial x_1}(u_1 - d_1) + \ldots + \frac{\partial J_{T_f}^*(x,t)}{\partial x_n}(u_n - d_n) \right\} \quad \text{(A1.2)}$$

Now, note that at each time instant *t*, if we knew $J_{T_f}^*(x,t)$ we would solve a linear programming problem for which $u_1, u_2, u_3, \ldots, u_n$ are the decision variables,

$\frac{\partial J_{T_f}^*}{\partial x_1}, \frac{\partial J_{T_f}^*}{\partial x_2}, \frac{\partial J_{T_f}^*}{\partial x_3}, \ldots, \frac{\partial J_{T_f}^*}{\partial x_n}$ are cost efficient and $\Omega(1,0,0,\ldots,0)$ is the constraint set

$$\Omega(1,0,0,\ldots,0) = \{(u_1, u_2, \ldots, u_n) \mid 0 \le U_1, u_2 = 0, u_3 = 0, \ldots, u_n = 0\} \quad \text{(A1.3)}$$

which is bound and convex.

It is known that the solution of the above linear programming is always at a vertex of the constraint set $\Omega(1,0,0,\ldots,0)$. That is $(u_1^*, u_2^*, u_3^*, \ldots, u_n^*)$ is either equal to $(0,0,0,\ldots,0)$

(if $\dfrac{\partial J^*_{T_f}}{\partial x_1}$ > 0) or equal to $(U_1,0,0,...,0)$ (if $\dfrac{\partial J^*_{T_f}}{\partial x_1}$ <0).

Furthermore, the solution is unique if the cost coefficient $\dfrac{\partial J^*_{T_f}}{\partial x_1}$ is non-zero. In the case of $\dfrac{\partial J^*_{T_f}}{\partial x_1}=0$, the solution is not unique anymore since any solution $(u_1^*,u_2^*,u_3^*,...,u_n^*)$ will not affect the objective function of the linear programming problem at the instant time *t*. however, to keep the cost efficient $\dfrac{\partial J^*_{T_f}}{\partial x_1}$ equal to zero at time $t+\delta t$, product Type 1 should be produced at the demand rate d_1 so as to minimize the rate of increase of the cost function $J^*_{T_f}$. In this case, $(u_1^*,u_2^*,u_3^*,...,u_n^*)$ is equal to $(d_1,0,0,...,0)$. A similar argument is used when the optimal setup state is $\sigma=(0,1,0,0,0,0,...,0)$.

Steady State Case: In this case, Let $t_s = t$ and $T_f = t_f$ in the expression of $J^*_{T_f}$ above. Let $J^* = \min_{\sigma,u\in\psi(\phi,\Omega)}\left\{ g(x,\sigma)+\dfrac{\partial J^*_{T_f}(x,t)}{\partial x_1}(u_1-d_1)+...+\dfrac{\partial J^*_{T_f}(x,t)}{\partial x_n}(u_n-d_n)\right\}$

(A1.4)

where

$V(x,t)=\lim_{\substack{(t_f-t_s)\\ t_f\to T_f\\ t_s\to t}} J^*_{T_f}-(t_f-t)J^*$.

Now, note that at each time instant *t*, if we knew $J^*_{T_f}(x,t)$ we would solve a linear programming problem for which $u_1,u_2,u_3,...,u_n$ are the decision variables,

$\partial V/\partial x_1, \partial V/\partial x_2, \partial V/\partial x_3,...,\partial V/\partial x_n$ are cost coefficients and $\Omega(1,0,0,...,0)$ is the constraint set.

$$\Omega(1,0,0,...,0)=\{(u_1,u_2,u_3,...,u_n)\mid 0\le u_1, u_2=0, u_3=0,...,u_n=0\} \quad \text{(A1.5)}$$

which is bounded and convex.

It is known that the solution of the above linear programming is always at a vertex of the constraint set $\Omega(1,0,0,...,0)$. That is $(u^*{}_1,u^*{}_2,u^*{}_3,...,u^*{}_n)$ is either equal to $(0,0,0,...,0)$ if $\partial V/\partial x_1$ > 0 or equal to $(U_1,0,0,...,0)$ (if $\partial V/\partial x_1<0$).

Furthermore, the solution is unique if the cost coefficient $\partial V/\partial x_1$ is non-zero. In the case of $\partial V/\partial x_1$=0, the solution is not unique anymore since any solution $(u^*{}_1,u^*{}_2,u^*{}_3,...,u^*{}_n)$ will not affect the objective function of the linear programming problem at the instant time *t,* however, to keep the cost efficient $\partial V/\partial x_1$ equal to zero at time $t+\delta t$, product Type 1 should be produced at the demand rate d_1 so as to minimize the rate of increase of the cost function $J^*_{T_f}$. In this case, $(u^*{}_1, u^*{}_2, u^*{}_3, \ldots, u^*{}_n)$ is equal to $(d_1, 0, 0, \ldots, 0)$. A similar argument is used when the optimal setup state is $\sigma = (0,1,0,0,0,0,0,0,\ldots,0)$.

APPENDIX A2

Proof of Theorem 2

This proof is based on the Karush-Kuhn-Tucker (KKT) optimality conditions. It is not difficult to show that the optimal $\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n$ can be obtained by solving the following non linear optimization problem:

$$F(\Upsilon_1,\Upsilon_2,\Upsilon_3,\ldots,\Upsilon_n)=\frac{(k+H_1(T_0+(\alpha_1-1)\Upsilon_1+\alpha_3\Upsilon_3)^2/2+H_2(T_0+\alpha_1\Upsilon_1+(\alpha_2-1)\Upsilon_2)^2/2+\ldots+H_n(T_0+\alpha_{n-1}\Upsilon_{n-1}+(\alpha_n-1)\Upsilon_n)^2/2}{T_0+\alpha_1\Upsilon_1+\alpha_2\Upsilon_2+\alpha_3\Upsilon_3+\ldots+\alpha_n\Upsilon_n}$$

subject to $\Upsilon_i \geq 0,\ i = 1, 2, 3, \ldots, n$

where

$H_i = \gamma_i d_i (1-\ell_i),\ i = 1, 2, 3, \ldots, n;$

$\alpha_i = (1-\ell_i)/(1-\ell),\ i = 1, 2, 3, \ldots, n;$

$T_0 = \delta/(1-\ell)$

Also it can be shown (see Elhafsi and Bai, 1996a; and Onanaye 2006) that $F(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n)$

is strictly convex in $\Upsilon_1, \Upsilon_2, \Upsilon_3, \ldots, \Upsilon_n$. Since the constraint domain is convex, it follows that KKT optimality conditions are given as follows:

$\nabla F(\Upsilon_1^*, \Upsilon_2^*, \ldots, \Upsilon_n^*) + u_1 \nabla g_1(\Upsilon_1^*, \Upsilon_2^*, \ldots, \Upsilon_n^*) + u_2 \nabla g_2(\Upsilon_1^*, \Upsilon_2^*, \ldots, \Upsilon_n^*) + \ldots + u_n \nabla g_n(\Upsilon_1^*, \Upsilon_2^*, \ldots, \Upsilon_n^*) = 0$

$u_1 \nabla g_1(\Upsilon_1^*, \Upsilon_2^*, \ldots, \Upsilon_n^*) = 0$

$u_2 \nabla g_2(\Upsilon_1^*, \Upsilon_2^*, \ldots, \Upsilon_n^*) = 0$

.

.

.

$u_n \nabla g_n(\Upsilon_1^*, \Upsilon_2^*, \ldots, \Upsilon_n^*) = 0$

$u_i \geq 0\ for\, i = 1, 2, 3, \ldots, n$

where Υ_i^* is the optimal value of $\Upsilon_i^* (i = 1, 2, 3, \ldots, n)$.

$\nabla F(\Upsilon_1^*, \Upsilon_2^*, \ldots, \Upsilon_n^*), \nabla g_1(\Upsilon_1^*, \Upsilon_2^*, \ldots, \Upsilon_n^*), \nabla g_2(\Upsilon_1^*, \Upsilon_2^*, \ldots, \Upsilon_n^*), \ldots, and\, \nabla g_n(\Upsilon_1^*, \Upsilon_2^*, \ldots, \Upsilon_n^*)$ are gradients of

$F(\Upsilon_1^*, \Upsilon_2^*, \ldots, \Upsilon_n^*), g_1(\Upsilon_1^*, \Upsilon_2^*, \ldots, \Upsilon_n^*), g_2(\Upsilon_1^*, \Upsilon_2^*, \ldots, \Upsilon_n^*), \ldots, and\, g_n(\Upsilon_1^*, \Upsilon_2^*, \ldots, \Upsilon_n^*)$ respectively.

Now, Letting $\Upsilon_i^* = 0 (i = 1, 2, 3, \ldots, n)$, calculating the gradient and substituting in the KKT conditions, we get:

$(\alpha_1/2-1)H_1+\alpha_2 H_3/2-\alpha_1 k/T_0^2-u_1=0;$
$\alpha_2 H_1/2+(\alpha_2/2-1)H_2-\alpha_2 k/T_0^2-u_2=0,$
.
.
.
$\alpha_n H_1/2+(\alpha_n/2-1)H_2-\alpha_n k/T_0^2-u_n=0,$

For the KKT conditions to hold true we must have
$u_i \geq 0 (i=1,2,3,...,n)$
.

Hence it follows that
$\alpha_1 k/T_0^2 \leq (\alpha_1/2-1)H_1+\alpha_1 H_n/2;$
$\alpha_2 k/T_0^2 \leq (\alpha_2/2-1)H_2+\alpha_2 H_1/2;$
.
.
.
$\alpha_n k/T_0^2 \leq (\alpha_n/2-1)H_n+\alpha_3 H_n/2.$
.

Substituting $\alpha_1,\alpha_2,\alpha_3,...,\alpha_n, and\, H_1,H_2,H_3,...,H_n$ by their expressions gives:
$k \leq T_0^2((1-\ell_n)(\gamma_n d_n-\gamma_1 d_1)+\ell\gamma_1 d_1)/2;$
$k \leq T_0^2((\ell+\ell_{n-1}-1)\gamma_{n-1}d_{n-1}+(1-\ell_1)\gamma_1 d_1)/2;$
.
.
.
$k \leq T_0^2((\ell+\ell_{n-1}-1)\gamma_n d_n+(1-\ell_{n-1})\gamma_{n-1}d_{n-1})/2;$

Now letting $k=0$, the following conditions are obtained:
$(1-\ell_n)(\gamma_n d_n-\gamma_1 d_1)+\ell\gamma_1 d_1)\geq 0;\ (a)$
$(\ell+\ell_1-1)\gamma_2 d_2+(1-\ell_1)\gamma_1 d_1)\geq 0;\ (b)$
.
.
.
$(\ell+\ell_{n-1}-1)\gamma_n d_n+(1-\ell_{n-1})\gamma_{n-1}d_{n-1})\geq 0;\ (z)$

Note that condition (a) is always satisfied since $\gamma_n d_n \geq \gamma_{n-1}d_{n-1} \geq \gamma_{n-2}d_{n-2} \geq ... \geq \gamma_1 d_1$

For condition (b) , (c), …, (z), if
$\ell+\ell_1-1\geq 0, \ell+\ell_2-1\geq 0, \ell+\ell_3-1\geq 0,...\ell+\ell_{n-1}-1\geq 0,$ then it is always satisfied. If
$\ell+\ell_1-1<0,$
then it is satisfied only if

$$\gamma_n d_n / \gamma_{n-1} d_{n-1} \le \gamma_{n-1} d_{n-1} / \gamma_{n-2} d_{n-2} \le \ldots \le \gamma_2 d_2 / \gamma_1 d_1 \le (1-\ell_1)/(1-\ell-\ell_1) \le (1-\ell_2)/(1-\ell-\ell_2) \le \ldots$$
$$\le (1-\ell_{n-2})/(1-\ell-\ell_{n-2}) \le (1-\ell_{n-1})/(1-\ell-\ell_{n-1}),$$

which complete the proof.

APPENDIX A3

Proof of Theorem 3

This theorem is proved by contradiction. Assume that $\Upsilon_1^* > 0, \Upsilon_2^* > 0, \Upsilon_3^* > 0, ..., and\, \Upsilon_n^* > 0$,

then $Q_1^*, Q_2^*, Q_3^*, ..., and\, Q_n^*$ are by 5.2(a), 5.2 (b), 5.2 (c),…, and 5.2 (z) (see the main work in chapter four), where *k=0*. In this case, we have:

$Q_1^* = q_1 d_n \gamma_n / (\alpha_n d_1 \gamma_1 + \alpha_1 d_n \gamma_n)$

$Q_2^* = q_2 d_1 \gamma_1 / (\alpha_2 d_1 \gamma_1 + \alpha_1 d_2 \gamma_3)$

$Q_3^* = q_3 d_2 \gamma_2 / (\alpha_3 d_2 \gamma_2 + \alpha_2 d_3 \gamma_3)$

$Q_4^* = q_4 d_3 \gamma_3 / (\alpha_3 d_4 \gamma_4 + \alpha_4 d_3 \gamma_3)$

.

.

.

$Q_n^* = q_n d_{n-1} \gamma_{n-1} / (\alpha_{n-1} d_n \gamma_n + \alpha_n d_{n-1} \gamma_{n-1})$

Now, using (5.2a), (5.2b), (5.2c), …, and (5.2z), we get

$\Upsilon_1^* = 2q_n d_{n-1}\gamma_{n-1} / d_n(\alpha_n d_{n-1}\gamma_{n-1} + \alpha_{n-1} d_n \gamma_n) - 2q_1 d_n \gamma_n \ell_1 / d_1(1-\ell_1)(\alpha_n d_1 \gamma_1 + \alpha_1 d_n \gamma_n) - \delta > 0;$

$\Upsilon_2^* = 2q_1 d_3\gamma_3 / d_3(\alpha_3 d_2\gamma_2 + \alpha_2 d_3 \gamma_3) - 2q_2 d_1 \gamma_1 \ell_2 / d_2(1-\ell_2)(\alpha_2 d_1 \gamma_1 + \alpha_1 d_2 \gamma_2) - \delta > 0;$

$\Upsilon_3^* = 2q_2 d_1\gamma_1 / d_1(\alpha_2 d_1\gamma_1 + \alpha_1 d_2 \gamma_2) - 2q_3 d_2 \gamma_2 \ell_3 / d_3(1-\ell_3)(\alpha_3 d_2 \gamma_2 + \alpha_2 d_3 \gamma_3) - \delta > 0;$

$\Upsilon_4^* = 2q_3 d_2\gamma_2 / d_3(\alpha_3 d_2\gamma_2 + \alpha_2 d_3 \gamma_3) - 2q_4 d_3 \gamma_3 \ell_4 / d_4(1-\ell_4)(\alpha_4 d_3 \gamma_3 + \alpha_3 d_4 \gamma_4) - \delta > 0;$

.

.

.

$\Upsilon_n^* = 2q_{n-1} d_{n-2}\gamma_{n-2} / d_{n-1}(\alpha_{n-1} d_{n-2}\gamma_{n-2} + \alpha_{n-2} d_{n-1} \gamma_{n-1}) - 2q_n d_{n-1} \gamma_{n-1} \ell_n / d_n(1-\ell_n)(\alpha_n d_{n-1} \gamma_{n-1} + \alpha_{n-1} d_n \gamma_n) - \delta > 0;$

substituting $q_1, q_2, q_3, ..., and\, q_n$ by their expressions, simplifying and rearranging terms gives:

$$
\begin{aligned}
&(1-\ell_n)d_1\gamma_1-(1+\ell_1)d_n\gamma_n>0;\\
&(1-\ell_1)d_2\gamma_2-(1+\ell_2)d_1\gamma_1>0;\\
&(1-\ell_2)d_3\gamma_3-(1+\ell_3)d_2\gamma_2>0;\\
&.\\
&.\\
&.\\
&and\\
&(1-\ell_{n-1})d_n\gamma_n-(1+\ell_n)d_{n-1}\gamma_{n-1}>0;\\
&d_n\gamma_n/d_1\gamma_1<(1-\ell_n)/(1+\ell_1)<1;\\
&d_2\gamma_2/d_1\gamma_1>(1-\ell_2)/(1+\ell_1)>1;\\
&d_3\gamma_3/d_2\gamma_2>(1-\ell_3)/(1+\ell_2)>1;\\
&.\\
&.\\
&.\\
&d_n\gamma_n/d_{n-1}\gamma_{n-1}>(1-\ell_n)/(1+\ell_{n-1})>1;
\end{aligned}
$$

which cannot be. Hence, in the case of no setup cost, $\Upsilon_1^*, \Upsilon_2^*, \Upsilon_3^*, ..., and\, \Upsilon_n^*$ cannot be non-zero simultaneously.

APPENDIX A4

Optimal Control for Initial Surplus Levels in Regions $R^o(R_1^o, R_2^o, R_3^o, ..., R_{[n(n-1)/2]}^o)$

To summarize the control actions in R_1^o.

* If $x \in T_1^{(1)}$:

1: setup the machine for product Type *1*;

2: After the surplus level of the product Type *1* at the rate U_1;

3: When the surplus level of product Type *1* becomes 0, change the production rate to d_1;

4: When the surplus level of product Type 2 becomes $x_2 = A_2^{(1)} + A_1^{(1)} d_2 \ell_1 / d_1(1-\ell_1)$, switch to the control actions of the cyclic schedule.

* If $x \in T_{11}^{(1)} \bigcup U_{11}^{(1)}$:

1: Do not produce either product Type:

2: When the surplus level of product Type *1* becomes $\delta_1 d_1$, start a setup change for product Type *1*.

3: After the setup change, produce product Type*1* at the demand rate d_1;

4: When the surplus level of product Type 2 becomes $x_2 = A_2^{(1)} + A_1^{(1)} d_2 \ell_1 / d_1(1-\ell_1)$ switch to the control actions of the cyclic schedule.

* If $x \in T_{12}^{(1)} \bigcup U_{21}^{(1)}$:

1: Do not produce either product Type;

2: When the surplus level of product Type 2 reaches levels $l_2^{(1)}$, *immediately* start a setup change for product Type 2.

$l_2^{(1)} = (1-\ell_2)(b^{(1)} - (d_2 / d_1) a^{(1)}) + (d_2 / d_1)(1-\ell_2) A_1^{(1)} + \ell_2 A_2^{(1)});$

3: at the end of the setup change, switch to the schedule control actions.

* If $x \in T_2^{(1)}$;

1: setup the machine for product Type 2;

2: After the setup change, produce product Type 2 at the rate of U_2 ;

3: When the surplus level of product Type 2 becomes 0, change the production rate to d_2;

4: When the surplus level of product type *1* becomes $x_1 = C_1^{(1)} + C_2^{(1)} d_1 \ell_2 (1-\ell_2)$, switch to the control actions of the cyclic schedule.

X = (a, b) be the initial vector of initial surplus in R_2^o

To summarize the control actions in R_2^o.

* If $x \in T_2^{(2)}$:

1: setup the machine for product Type 2;

2: After the surplus level of the product Type 2 at the rate U_2;

3: When the surplus level of product Type 2 becomes 0, change the production rate to d_2;

4: When the surplus level of product Type 3 becomes $x_3 = A_2^{(2)} + A_1^{(2)} d_3 \ell_2 / d_2 (1-\ell_2)$, switch to the control actions of the cyclic schedule.

* If $x \in T_{11}^{(2)} \cup U_{11}^{(2)}$:

1: Do not produce either product Type:

2: When the surplus level of product Type 2 becomes $\delta_2 d_2$, start a setup change for product Type 2.

3: After the setup change, produce product Type 2 at the demand rate d_2;

4: When the surplus level of product Type 3 becomes $x_3 = A_2^{(2)} + A_1^{(2)} d_3 \ell_2 / d_2 (1-\ell_2)$ switch to the control actions of the cyclic schedule.

* If $x \in T_{12}^{(2)} \cup U_{21}^{(2)}$:

1: Do not produce either product Type;

2: When the surplus level of product Type 2 reaches levels $l_2^{(2)}$, *immediately* start a setup change for product Type 3.

$l_2^{(2)}$ is given by

$l_2^{(2)} = (1-\ell_3)(b^{(2)} - (d_3/d_2)a^{(2)}) + (d_3/d_2)(1-\ell_3)A_1^{(2)} + \ell_3 A_2^{(2)});$

3: at the end of the setup change, switch to the schedule control actions.

* If $x \in T_2^{(3)}$;

1: setup the machine for product Type 3;

2: After the setup change, produce product Type 3 at the rate of U_3 ;

3: When the surplus level of product Type 3 becomes 0, change the production rate to d_3;

4: When the surplus level of product type 2 becomes $x_2 = C_1^{(2)} + C_2^{(2)} d_2 \ell_3 (1-\ell_3)$, switch to the control actions of the cyclic schedule.

X= (a, b) be the initial vector of initial surplus in R_3^o.

To summarize the control actions in R_3^o.

* If $x \in T_1^{(3)}$:

1: setup the machine for product Type *1*;

2: After the surplus level of the product Type *1* at the rate U_1;

3: When the surplus level of product Type *1* becomes 0, change the production rate to d_1;

4: When the surplus level of product Type 3 becomes

$x_3 = A_2^{(3)} + A_1^{(3)} d_3 \ell_1 / d_1(1-\ell_1)$, switch to the control actions of the cyclic schedule.

* If $x \in T_{11}^{(3)} \bigcup U_{11}^{(3)}$:

1: Do not produce either product Type:

2: When the surplus level of product Type *1* becomes $\delta_1 d_1$, start a setup change for product Type *1*.

3: After the setup change, produce product Type*1* at the demand rate d_1;

4: When the surplus level of product Type 3 becomes $x_3 = A_2^{(3)} + A_1^{(3)} d_3 \ell_1 / d_1(1-\ell_1)$ switch to the control actions of the cyclic schedule.

* If $x \in T_{12}^{(3)} \bigcup U_{21}^{(3)}$:

1: Do not produce either product Type;

2: When the surplus level of product Type 3 reaches levels $l_2^{(3)}$, *immediately* start a setup change for product Type 3.

$l_2^{(3)}$ is given by

$l_2^{(3)} = (1-\ell_3)(b^{(3)} - (d_3 / d_1) a^{(3)}) + (d_3 / d_1)(1-\ell_3) A_1^{(3)} + \ell_3 A_2^{(3)}$);

3: at the end of the setup change, switch to the schedule control actions.

* If $x \in T_2^{(3)}$;

1: setup the machine for product Type 3;

2: After the setup change, produce product Type 3 at the rate of U_3 ;

3: When the surplus level of product Type 3 becomes 0, change the production rate to d_3;

4: When the surplus level of product type *1* becomes

$x_1 = C_1^{(3)} + C_2^{(3)} d_1 \ell_3 (1-\ell_3)$, switch to the control actions of the cyclic schedule.

Thus we repeat this process for other regions until we get to region $R^o_{[n(n-1)/2]}$.

Let $x = (a,b)$ be the vector of initial surplus in $R^o_{[n(n-1)/2]}$.

* If $x \in T_n^{(r)}$:

1: setup the machine for product Type *n-1*;

2: After the surplus level of the product Type *n-1* at the rate U_n;

3: When the surplus level of product Type *n-1* becomes 0, change the production rate to d_{n-1};

4: When the surplus level of product Type n becomes $x_n = A_2^{(r)} + A_1^{(r)} d_n \ell_{n-1} / d_{n-1}(1-\ell_{n-1})$, switch to the control actions of the cyclic schedule.

* If $x \in T_{11}^{(r)} \cup U_{11}^{(r)}$:

1: Do not produce either product Type:

2: When the surplus level of product Type *n-1* becomes $\delta_{n-1} d_{n-1}$, start a setup change for product Type *n-1*.

3: After the setup change, produce product Type *n-1* at the demand rate d_{n-1};

4: When the surplus level of product Type n becomes $x_n = A_2^{(r)} + A_1^{(r)} d_n \ell_{n-1} / d_{n-1}(1-\ell_{n-1})$ switch to the control actions of the cyclic schedule.

* If $x \in T_{12}^{(r)} \cup U_{21}^{(r)}$:

1: Do not produce either product Type;

2: When the surplus level of product Type n reaches levels $l_n^{(r)}$, *immediately* start a setup change for product Type n.

$l_2^{(r)} = (1-\ell)(b^{(r)} - (d_n / d_{n-1})a^{(1)}) + (d_n / d_{n-1})(1-\ell_n)A_1^{(r)} + \ell_n A_2^{(r)})$;

3: at the end of the setup change, switch to the schedule control actions.

* If $x \in T_2^{(r)}$;

1: setup the machine for product Type n;

2: After the setup change, produce product Type n at the rate of U_n ;

3: When the surplus level of product Type n becomes 0, change the production rate to d_n;

4: When the surplus level of product type *n-1* becomes $x_{n-1} = C_1^{(r)} + C_2^{(r)} d_{n-1} \ell_n (1-\ell_n)$, switch to the control actions of the cyclic schedule.