QUADRATIC PROGRAMMING AS AN OPTIMIZATION TOOL FOR PORTFOLIO MANAGEMENT

Article · March 2016

2 authors:

Adeniyi Samson Onanaye
Redeemer's University

Peace Oluwalonimi Banjo

International Journal of Mathematics and
Computer Applications Research (IJMCAR)
ISSN(P 2249-6955; ISSN(E): 2249-8060
Vol. 6, Issue 2, Apr 2016, 19-32
TJPRC Pvt. Ltd.

QUADRATIC PROGRAMMING AS AN OPTIMIZATION TOOL FOR PORTFOLIO MANAGEMENT

ONANAYE, A.S. & AGBOLAMAGBIN, P. O

Mathematical Sciences Department, Redeemer's University, P.M.B. Ede, Osun State, Nigeria

ABSTRACT

In this paper, an investigation of Nigerian stocks was carried out. The principal task concerns essentially optimizing the portfolio or investors in Nigeria Stocks. Observations revealed that, though it is possible to get profits from investments in some securities but attached to them are some measure of risks. The paper has been able to provide a solution to the problem of balancing the profits and risks attached in the management of portfolios. The optimization tools used in this work includes the mean-variance principle which was used to model the problem and the extended simplex method which was employed in solving the problem.

KEYWORDS: Quadratic; Programming; Portfolio; Management; Optimization; and Karush-Kuhn-Tucker Conditions

2010 Mathematics Subject Classification: 12Y05

Received: Feb 20, 2016; **Accepted:** Mar 09, 2016; **Published:** Mar 16, 2016; **Paper Id.:** IJMCARAPR20163

Original Article

1.0 INTRODUCTION

It is fast becoming a major means of getting profit by investing money in different securities, namely bonds, equities, e.t.c. Though, it is possibly to get profits from the investments but attached to these securities is different measure of risk. Hence, it has become imperative that we find a solution to the problem of balancing the profit and risk in the portfolio.

Quadratic programming is a unique case of mathematical optimization problem. It is the problem of maximizing or minimizing a quadratic function where the variables are subject to linear constraints. Quadratic programming is used in portfolio optimization for the formulation of the mean-variance optimization of investments judgements under uncertainty. (Markowitz, 1959).

Portfolio management is a major aspect of economics and finance. Portfolio management, for an investor that has different assets to trade in, is choosing optimal investments, that is, how many shares of an asset should he/she buy and hold for him/her to maximise some criteria depending on his/her total wealth and/or consumption. The portfolio that meets this criterion is known as efficient portfolio. The theory of portfolio optimization was developed by a renowned economist, Harry Markowitz in the 1950's and he was rewarded with a Nobel Prize in Economics in 1990.

One of the major applications of quadratic programming is the portfolio selection problem. According to Fernando (2008), portfolio optimization is also known as the mean-variance optimization, where the mean is the expected value of the return of the investment and the variance is the measure of the risk involved in the portfolio. Now the two major competing goals of an investment is long-term increase in capital and low risk. The Markowitz model is an optimization model that specialises in balancing the risk and return of the portfolio. The main idea of

Markowitz's portfolio optimization problem is that the portfolio's return is a function of its risk.

The portfolio risk is measured from historical data while the portfolio return is the expected future return.

The major area of applied mathematics covered in this project is Financial Mathematics which is concerned with the formulation, validation and implementation of mathematical financial models. Modern portfolio theory allows the idea of diversification in investing, with the aim that selection of a collection of different assets has collectively lower risk than that of an individual asset.

1.2Quadratic Programming

A Quadratic programming problem is a problem where the objective is a quadratic function of the decision variables, and the constraints are linear functions of the variables. A quadratic programming problem in its standard form can be written as

$$\min_x \frac{1}{2} x^T Q x + c^T x$$

subjectto (4)

$$Ax = b$$

$$x \geq 0$$

According to Reha (Spring 2003), when Q is a positive semi definite matrix, then the quadratic programming problem is a convex function of x, which is true for portfolio optimization problems. QP problems have only one feasible region but the optimal solution may be found anywhere within the region.

A widely used QP problem is the Markowitz mean-variance portfolio optimization problem, in this case the quadratic objective function is the portfolio variance and the linear constraints specify a lower bound for portfolio return. According to Fernando (2008), there are various formulations of the portfolio optimization problems:

- Maximizing expected return for a given level of risk.
- Minimizing risk for a given value of expected return.
- Minimizing risk and maximizing expected returns using a specified risk aversion factor.
- Minimizing risk irrespective of expected returns.
- Maximizing expected return irrespective of risk.

The above scenarios may have equality and inequality constraints, linear or nonlinear constraints and so different methods can be used to solve them.

1.3 Problem Statement

In times past, investors have majorly used their reasoning and other approaches in the selection and management of their portfolio and later realised that mistakes have been made. This study is about how scientific approach, that is, quadratic programming can be used to avoid such mistakes and help in optimal portfolio management. The case study to be used in implementation of this work is the Nigeria Stock Exchange.

1.4 The Quadratic Model

Let us assume that an investor has ***n*** different **assets**. The **rate of return** on each of the asset, say ***i***, ***i***= *1*,...,***n***, is a random variable with expected value m_i. The problem of the investor is to know what fraction of total funds $x_i$he should invest in each asset ***i*** for him to minimize risk, for a given minimum expected rate of return.

Let ***F*** denotes the covariance matrix of rates of assets returns. The mean-variance model consists of minimizing portfolio risk which is given as,

$$\frac{1}{2}x^T F x \tag{5}$$

which is subject to a set of constraints.

The expected returns should be greater than a minimal rate of return **r**that the investor wants,

$$\sum_{i=1}^{n} m_i x_i \geq r \tag{6}$$

The portfolio vector **x**must satisfy,

$$\sum_{i=1}^{n} x_i = 1 \tag{7}$$

The optimization problem becomes a quadratic programming problem, because the objective function is quadratic, and the constraint is linear,

(Source: Math Works)

2.0 BASIC CONCEPT OF PORTFOLIO OPTIMIZATION

However important the linear programming model is, it has its limits. This is where quadratic programming comes in, since not all physical phenomenons are linear. The quadratic model allows the objective function to be quadratic. (Gerard, 2005).

The sharing of capital over different investment choices is called a portfolio. (Elke and Ralf Korn, 2007)

The portfolio selection problem can be devised as follows;

$$\text{Min } X^TQX=V(r_p)$$

$$\max \mu^TX=E(r_p) \tag{8}$$

$$\text{s.t. } Ax \leq b,\ 1^TX= 1$$

$$L \leq x \leq U$$

Where X is the share on individual assets, Q is the covariance matrix of rates of returns, $V(r_p)$ is the measure of the portfolio risk, $E(r_p)$ is the expected portfolio returnand l is the lower bound, u is the upper bound.

This model has two major objectives as shown by Markowitz, (1959). Two main objectives are general to investors,

- They want the portfolio return to be high.
- They want this return to be steady and not subject to uncertainty.

(Marcus and Ralph, 2008).

We also have other formulations of the portfolio optimization problem, for example Bodie, Kane and Marcus (2004) looks at it by,

$$\begin{aligned} &\text{Min } X^TQX \\ &\text{s.t. } \mu^TX = e \\ &Ax \leq b,\ x \geq 0 \end{aligned} \tag{9}$$

Where e is the specified expected return. The idea is to minimize risk for a given expected return

We can also maximize the expected return for a specified risk.

$$\begin{aligned} &\max \mu^TX \\ &\text{s.t. } X^TQX \leq m \\ &\textstyle\sum_{i=1}^{n} X_i = 1 \end{aligned} \tag{10}$$

Where m is the specified maximum portfolio risk. (Anna N., 2009)

Another formulation is the Markowitz model,

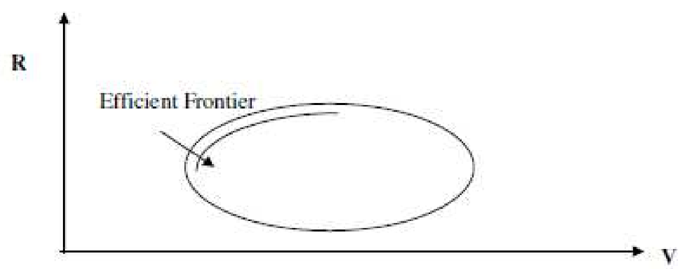
$$\begin{aligned} &\max \alpha R - (1-\alpha)V \\ &\text{s.t. } x_i \geq 0,\ i = 1,\dots, n \\ &\textstyle\sum_{i=1}^{n} X_i = 1 \end{aligned} \tag{11}$$

Where α is a risk-aversion factor, R is the total return on each asset and V is the variance. (Markowitz, 1959).

Markowitz's work suggested that the choice between risk and return differs for eachinvestor, butthe preferences of all people lie upon a false curve, which is usually called the "efficientfrontier''.

2.1 Efficient Frontier

Figure 2.1: Efficient Frontier



(**Source:** Anna Nagurney.Portfolio Optimization.2009)

An efficient frontier is a portfolio that is mean-variance efficient. That is, the portfolio minimizes risk for a given expected return and maximizes return for a specified risk. (Kroll and Levy, 1984).

For instance, assume we have two securities, A and B, with the following risk-return values.

	A	B	C
1		**Security A**	**Security B**
2	**Expected Returns**	12%	20%
3	**Standard Deviation**	20%	40%
4			
5	**Correlation**	-0.2	

Figure 2

We combine these two a securities to create a portfolio. In the portfolio, we can combine the two securities with different weights for each asset to create a wide range of portfolios having different risk-return values. For example, if we take 50% of each asset, the expected return and risk of the portfolio is given as follows,

E(R) = 0.50*12% + 0.50*20% = 16%

σ = Sqrt (0.20^2*0.5^2+0.40^2*0.5^2+2*(-0.2)*0.5*0.5*0.2*0.4) = 20.49%

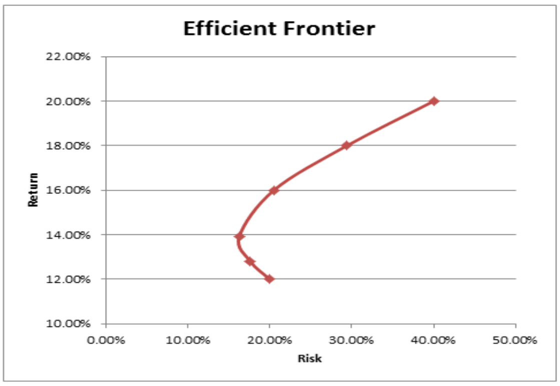
The table below gives the risk-return values for the different portfolios created by combining the two securities with different weights.

Portfolio	A Proportion	B Proportion	Expected Return	Standard Deviation
1	1	0	12.00%	20.00%
2	0.9	0.1	12.80%	17.64%
3	0.76	0.24	13.92%	16.27%
4	0.5	0.5	16.00%	20.49%
5	0.25	0.75	18.00%	29.41%
6	0	1	20.00%	40.00%

Figure 3

The return and risk of the portfolios created can be plotted with return on Y-axis and risk on X-axis. The graph is given belowand is called the efficient frontier.

Figure 4



(**Source:** Finance Train)

A portfolio that meets all the specified constraints is known as a *feasible portfolio*. (Kenneth, 2005).

The reduction in the variation of the rate of return by sharing the capital over different investment assets is known as the diversification effect. (Ralf Korn, 2007).

2.3 Computational Improvements in Portfolio Optimization

The Markowitz's model has been used since some decades ago as the basis for the development of the modern portfolio theory. However, different from its hypothetical views, it is not used to constructlarge-scale portfolios. (Konno, 1991). The most important reason for this is the difficulty of solving a large-scale quadratic programming problem with a dense covariance matrix during computation. (Yamazaki, 1991).

3.0 METHODOLOGY

3.1 A General Optimization Problem

In general, the optimization problem is formulated as a function **f** of n variables x= (x_1,...,x_n) written as

f (x) = f (x_1,..., x_n)

This function f (x_1,...,x_n) is called the objective function and the variables x= (x_1,..., x_n) are called decision variables.

Optimization Constraints

Theorem 1: Concave and Convex function

If a function say, f(x) has second order derivatives in [a, b],

- Then f(x) is convex on the interval [a, b], if $f''(x) \geq 0$, for all x. We say that f(x) is strictly convex if, $f''(x) > 0$.
- Similarly, f(x) is concave if, $f''(x) \leq 0$, for all x. Also f(x) is strictly concave if,

  $f''(x) < 0$.

  (proof omitted)

3.2 KKT Optimality Conditions

Theorem 2

Assuming that x is a local minimum of the QP problem,

$\text{Min}\frac{1}{2}x^TQx + c^Tx$

Ax=b (3a)

$x_i \geq 0$for all i

So that it satisfies the constraints and that Q is a positive definite matrix. Then we have vector u and v such that,

$A^Tu - Qx - v = c$

$v \geq 0$ (3b)

$x_iv_i = 0, \forall i$

Therefore, x is a global solution. (Proof omitted)

Definition 1

The Karush-Kuhn-Tucker conditions given above is a collection of conditions for,

- **Primal feasibility:** $Ax=b$, $x \geq 0$
- **Dual feasibility:** $ATu - Qx - v = c$, $v \geq 0$
- **Complementary slackness:** $x_i v_i = 0, \forall i$

3.3 Extension of Simplex Method

Now for the **extended simplex method**,we did the following.

- Construct the main KKT conditions for non-negative variables for the problem.
- Add slack and surplus variables for the inequalities.
- Add artificial variables for any equation that does not have obvious basic variables.
- Use the following modifications;
  - Never pivot such that the surplus variable and the basic variable both become basic.
  - Never pivot such that the surplus/slack variable and the KKT multiplier both become basic.

To construct the KKT conditions for the problem we have,

The Karush-Kuhn-Tucker conditions given as:

- **Primal Feasibility:** $Ax=b$, $x \geq 0$
- **Dual Feasibility:** $A^T u - Qx - v = c$, $v \geq 0$
- **Complementary Slackness:** $x_i v_i = 0, \forall i$

4.0 APPLICATION OF QUADRATIC PROGRAMMING USING CASE STUDY

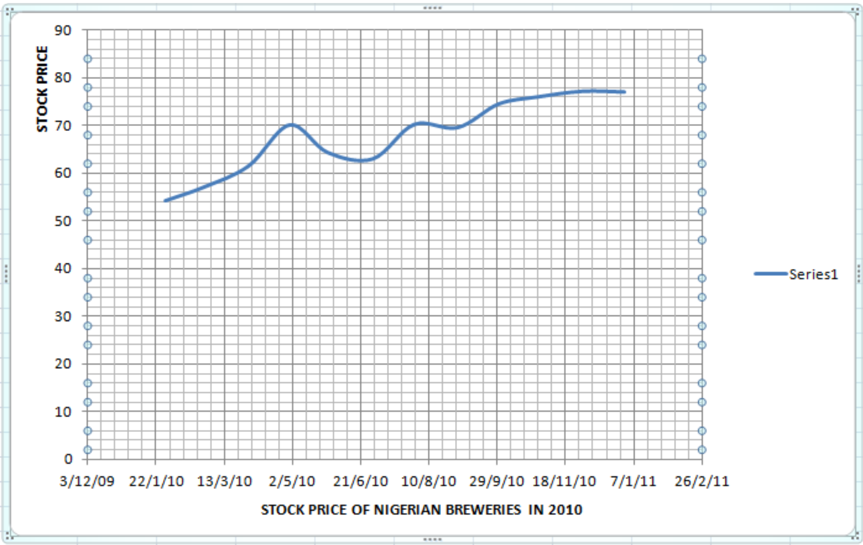
4.1 Research Case Study

In this work, we used two assets namely, FIRST BANK NIGERIA PLC, NIGERIA BREWERIES stocks. The expected rate of return and the risk is given below,

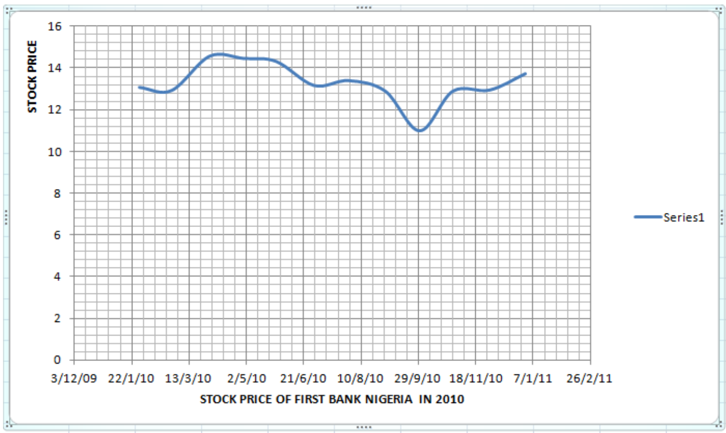
Table 4.1: Annual Rate of Returns and Risk for First Bank, Nig. Breweries Stocks

	First bank (FB)	Nigeria breweries(NB)
Expected returns	20.7%	20.6%
Variance	0.008%	0.012%
Covariance	0.0067	

The values in the table above are results from the daily stock prices from 2010-2013.



Graph 4.1: Stock Price of Nigeria. Breweries. In 2010-2011



Graph 4.1: Stock Price of First Bank Nigeria in 2010-2011

Let's assume we have capital K that is to be invested and we have two assets A and B. We also invest an amount x_1 in asset A and invest x_2 in asset B. Let's denote x_1/K,which is the share of the capital invested in asset A, by y_1 and denote x_2/K,which is the share of the capital invested in asset B, by y_2. The asset A has expected rate of return r_1%, also asset B has expected rate of return r_2%. Then we have the total returns r^*%from investment after a year as,

$$\frac{r^*}{100} = y_1 \frac{r_1}{100} + y_2 \frac{r_2}{100} \text{ (a)}$$

$$y_1 + y_2 = 1$$

$$y_1, y_2 \geq 0$$

We have the total variance,

$$Var(y_1.yield_1 + y_2.yield_2) = y_1^2.Var(yield_1) + y_2^2.Var(yield_2) + 2.y_1.y_2Cov(yield_1, yield_2)\ (b)$$

Let's assume that we have a capital of ₦ 100,000 and the minimum total returns we want from investing in both assets is 23%. Then we use the **mean-variance principle** to minimize the risk subject to the maximum expected returns. Using Tab 4.1, we have the total returns as,

$$r^* = y_1 * 20.7 + y_2 * 20.6 \text{ (c)}$$

We also have total variance as,

$$y_1^2 * 0.007 + y_2^2 * 0.007 + 2.y_1.y_2 * 0.0067 \; (d)$$

So we have,

$$\min \; 0.02y_1^2 + 0.01y_2^2 + 0.01y_1$$

$$subject\ to\ 20.7y_1 + 20.6\,y_2 \le 23$$

$$y_1 + y_2 = 1$$

$$y_1, y_2 \ge 0$$

Note that y_1 is the share invested in First Bank Stock and y_2 is the share invested in Nigeria Breweries.

The above minimization problem is a quadratic programming problem and we use the extended simplex method to solve the problem.

$$\max \; -0.02y_1^2 - 0.01y_2^2 - 0.01y_1 = z$$

$$subject\ to\ 20.7y_1 + 20.6\,y_2 \le 23$$

$$y_1 + y_2 = 1$$

$$y_1, y_2 \ge 0$$

The problem is converted to a maximization problem by letting min z to be max –z.

We first find if the given function is concave or convex.

$$\frac{\partial z}{\partial y_1} = -0.04y_1 - 0.01$$

$$\frac{\partial z}{\partial y_2} = -0.02y_2$$

$$\frac{\partial^2 z}{\partial y_1^2} = -0.04$$

$$\frac{\partial^2 z}{\partial y_2^2} = -0.02$$

$$\frac{\partial^2 z}{\partial y_1 \partial y_2} = 0$$

$$H(y) = \begin{vmatrix} \frac{\partial^2 z}{\partial y_1^2} & \frac{\partial^2 z}{\partial y_1 \partial y_2} \\ \frac{\partial^2 z}{\partial y_1 \partial y_2} & \frac{\partial^2 z}{\partial y_2^2} \end{vmatrix}$$

$$H(y) = \begin{vmatrix} -0.04 & 0 \\ 0 & -0.02 \end{vmatrix}$$

$$I = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$\lambda(I) = \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix}$$

$$H(y) - \lambda(I) = \begin{vmatrix} -0.04 - \lambda & 0 \\ 0 & -0.02 - \lambda \end{vmatrix}$$

Determinant of $H(y) - \lambda(I) = 0$

$$(-0.04 - \lambda)(-0.02 - \lambda) = 0$$

$$\lambda(\lambda + 0.04) + 0.02(\lambda + 0.04) = 0$$

$$\lambda = -0.04, -0.02$$

Therefore the function is concave since λ is negative.

$$\theta(y, \lambda) = -0.02y_1^2 - 0.01y_2^2 - 0.01y_1 + \lambda_1(23 - 20.7y_1 - 20.6\,\mathrm{y}_2) + \lambda_2(1 - y_1 - \mathrm{y}_2) \ *$$

Differentiate * w.r.t $y_1$and y_2

$$\theta y_1 = -0.01 - 0.04y_1 - 20.7\lambda_1 - \lambda_2 = -\mu_1 \ (a)$$

$$\theta y_2 = -0.02y_2 - 20.6\lambda_1 - \lambda_2 = -\mu_2 \ (b)$$

Where $\mu_1$and μ_2 are surplus variables.

$$y_1, y_2, \lambda_1, \lambda_2, \mu_1, \mu_2 \geq 0$$

$$\mu_1 x_1 = 0, \mu_2 x_2 = 0$$

Differentiate w.r.t λ_1, λ_2

$$\theta\lambda_1 = 23 - 20.7y_1 - 20.6\,\mathrm{y}_2 = S_1 \ (c)$$

$$\theta\lambda_2 = 1 - y_1 - \mathrm{y}_2 = S_2 \ (d)$$

Where S_1, S_2 are slack variables.

$$\lambda_1 S_1 = 0, \ \lambda_2 S_2 = 0$$

4.2 KKT Conditions

From equations a, b, c and d, we get.

$$0.04y_1 + 20.7\lambda_1 + \lambda_2 - \mu_1 = -0.01$$

$$0.02y_2 + 20.6\lambda_1 + \lambda_2 - \mu_2 = 0$$

$$20.7y_1 + 20.6\,\mathrm{y}_2 + S_1 = 23$$

$$y_1 + \mathrm{y}_2 + S_2 = 1$$

Introducing artificial variables A_1 and A_2 to the first two equations to obtain a feasible solution of the problem.

$$\max \ -A_1 - A_2$$

$$subject\ to\ 0.04y_1 + 20.7\lambda_1 + \lambda_2 - \mu_1 = -0.01$$

$0.02y_2 + 20.6\lambda_1 + \lambda_2 - \mu_2 = 0$

$20.7y_1 + 20.6\,y_2 + S_1 = 23$

$y_1 + y_2 + S_2 = 1$

$y_1, y_2, \lambda_1, \lambda_2, \mu_1, \mu_2, S_1, S_2 \geq 0$

Now we solve the problem using Simplex Method:

Table 4.2: First Tableau

	C_j	0	0	0	0	0	0	0	0	-1	-1	
C_B	Basic variable B	y_1	y_2	μ_1	μ_2	S_1	S_2	λ_1	λ_2	A_1	A_2	Solution values b=X_B
-1	A_1	0.04	0	-1	0	0	0	20.7	1	-1	0	0.01
-1	A_2	0	0.02	0	-1	0	0	20.6	1	0	-1	0
0	S_1	20.7	20.6	0	0	1	0	0	0	0	0	23
0	S_2	1	1	0	0	0	1	0	0	0	0	1
$Z_j - C_j$		-0.04	-0.02	1	1	0	0	-40.1	-2	0	0	

If we make sure that the pair (λ_1, S_1) and (μ_1, x_1) are not basic at the same time then the conditions $\lambda_i S_i = 0$, and $\mu_1 x_1 = 0$ will be satisfied.

Although we have that $Z_j - C_j$ is smallest in the columns λ_1, λ_2 they cannot be made basic variables since S_1, S_2 are already basic variables. Therefore y_1 enters and A_1 leaves.

Table 4.3: Second Tableau

	C_j	0	0	0	0	0	0	0	0	-1	
C_B	Basic variable B	y_1	y_2	μ_1	μ_2	S_1	S_2	λ_1	λ_2	A_2	Solution Values b=X_B
0	y_1	1	0	-25	0	0	0	517.5	25	0	0.25
-1	A_2	0	0.02	0	-1	0	0	20.6	1	-1	0
0	S_1	0	20.6	517.5	0	1	0	8869.7	-517.5	0	17.825
0	S_2	0	1	25	0	0	0	-517.5	-25	0	0.75
$Z_j - C_j$		0	-0.02	0	1	0	0	-20.6	-1	0	

Although we have that $Z_j - C_j$ is smallest in the columns λ_1 it cannot be made basic variables since S_1 is already a basic variable. Therefore y_2 enters and S_2 leaves.

Table 4.4: Final Tableau

	C_j	0	0	0	0	0	0	0	-1	
C_B	Basic Variable B	y_1	y_2	μ_1	μ_2	S_1	λ_1	λ_2	A_2	Solution values b=X_B
0	y_1	1	0	-25	0	0	517.5	25	0.25	0.25
-1	A_2	0	0	-0.5	-1	0	-30.9	-1.5	-1	37.5
0	S_1	0	0	2.5	0	1	-1790.8	-2.5	0	2.34
0	y_2	0	1	25	0	0	0	-517.5	0	0.75
$Z_j - C_j$		0	0	0.5	1	0	30.9	1.5	1	

Since all elements in the $Z_j - C_j$ row are non-negative, therefore the current basic solution is the optimal solution.

The values for y_1 and y_2 are 0.25 and 0.75 respectively. The associated objective function is 0.009%.

4.3Discussions of Solution

From the problem solved, we see that for the investor to have a minimum rate of return of 23% he has to invest a share of 0.75 in the First Bank stock and a share of 0.25 in the Nigeria Breweries Stock. The rate of return of the portfolio is subject to 0.009% risk. The Simplex Method is the simplest method to solve a Quadratic programming problem.

5.0 CONCLUSIONS

In conclusion, the quadratic programming methods is the simplest and most efficient way of optimizing an investment portfolio. In the theoretical or practical standpoint, the most efficient way of optimizing a portfolio is the **diversification effect**, which is the reduction of fluctuation of the rate of return by distributing the capital over different securities.

REFRENCES

1. *Anna Nagurney,Portfolio Optimization (2009), [Electronic Version], Retrieved on May, 2013 from supernet.isenberg.umass.edu.*
2. *AvinashSathaye. Understanding the Simplex algorithm.(2009), [Electronic Version], Retrieved on May, 2013 from www.ms.uky.edu*
3. *C. Van De Panne, A. Whinston, The Simplex and the Dual Method for Quadratic Programming,Operations Research Quarterly, Vol.15, 1964, pp.355-388.*
4. *Elizabeth Wong, Active -Set Methods for Quadratic Programming (2011), [Electronic Version], Retrieved on May, 2013 from www.researchgate.net*
5. *E. Korn,R.Korn, Portfolio-optimization by the mean-variance-approach (2008), [Electronic Version], Retrieved on May, 2013 from http://www.mathematik.uni- kl.de/~mamaeusch/*
6. *G. Fernando, MATH 5750/6880 TOPICS IN OPTIMIZATION FALL 2008 SYLLABUSwww.math.utah.edu/~fguevara/math5750_f08/syllabus.pdf*
7. *GdiTollo, A. Roli,International Journal of Operations Research (2008,MetaheuristicsFor The Portfolio Selection Problem.*
8. *G. Cornuejols, R. Tutuncu, Optimization Methods in Finance (2005), [Electronic Version], Retrieved on May, 2013 from www.cambridge.org/us/...**finance**/**optimization-methods-finance***
9. *G. Gordon, R. Tibshirani. Karush-Kuhn-Tucker conditions. (2006), [Electronic Version], Retrieved on May, 2013 from https://www.cs.cmu.edu/~**ggordon**/10725-F12/slides/16-**kkt**.pdf*
10. *J. Larrosa, A. Oliveras,Mixed Integer Programming.), [Electronic Version], Retrieved on May, 2013 from http://www.lsi.upc.edu/~erodi/webpage/rpar/56.pdf*
11. *Javier Nogales,Optimization Models and Methods with Applications in Finance (2013)), [Electronic Version], Retrieved on May, 2013 from*
12. *www.bcamath.org/documentos_public/.../Nogales*

13. J. Reeb, S. Leavengood,Using the Simplex Method to Solve Linear Programming Maximization Problems (2008).Operations Research.
14. M. Capinski, T. Zastawniak. Mathematics for Finance, An Introduction to Financial Engineering (2003).Published by Springer-Verlag London Limited.
15. Mark R. Greene. Application of Mathematics to Insurance and Risk Management. The Journal of Insurance, Vol. 28, No. 1 (1961), pp. 93-104. Published by American Risk and Insurance Association.
16. Martina Vankova,Algorithms for the Solution of the Quadratic Programming Problem (2004), [Electronic Version], Retrieved on May, 2013 from
17. N.I. Gould, P.L. Toint,A Quadratic Programming Bibliography (2001).RAL Numerical Analysis Group Internal Report 2000–1.
18. Philip Wolfe,The Simplex Method for Quadratic Programming. Econometrica, Vol. 27, No. 3, (Jul., 1959), pp. 382-398. Published by The Econometric Society
19. R. Cottle, G. Infanger. HarryMarkowitz and the Early History of Quadratic Programming (2010).Published by Springer Science And Business Media, LLC.
20. R.J. Vanderbei, D.F. Shanno, An Interior Point Algorithm forNon-convex Nonlinear Programming (1999), [Electronic Version], Retrieved on May, 2013 from www.princeton.edu/~rvdb/ps/predcor.**pdf**
21. Robert M. Freund,Solution Methods for Quadratic Optimization (2004), [Electronic Version], Retrieved on May, 2013 from
22. ocw.mit.edu/courses/...**optimization**...**2004**/.../14solving_qp_art.pdf
23. R.S. Takasaki,Socially responsible investing: A Mean-variance Analysis (2011), [Electronic Version], Retrieved on May, 2013 from www.politesi.polimi.it
24. RehaTutunci,Optimization in finance (2003), Advanced Lecture on Mathematical Science and Information Science I.
25. S. Tungsong, G. Srijuntongsir, Applications of conic optimization and quadratic programming in the Thai derivatives and equity markets. International science index (2010),Vol 42,pp 1069-1071.
26. S. Vavasis, P. Panos. Quadratic Programming With One Negative Eigenvalue Is NP-Hard (1991).Journal of Global Optimization 1.pp 15-22.
27. V. Moraru,An Algorithm for Solving Quadratic Programming Problems. Computer Science Journal of Moldova, vol.5, No.2 (14), 1997
28. W.L. Winston, Introduction to mathematical Programming, (1995).Published by Duxbury Press.
29. Xin-She Yang.Metaheuristic Optimization (2011), [Electronic Version], Retrieved on May, 2013 from www.scholarpedia.org/article/**Metaheuristic_Optimization**